Dynamics of theta sums
Krakow summer school

Matthew Welsh

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Contents

1 Classical results

In the first sections, we work entirely work in the *rank* one setting. That is, theta sums with one variable of summation. These have the form

$$\theta_f(N,\alpha,x,y) = \sum_{n\in\mathbb{Z}} f(\frac{1}{N}(n+x))\mathrm{e}(\frac{1}{2}\alpha(n+x)^2 + ny), \tag{1}$$

where f is a function with sufficiently rapid decay. Often f will be the indicator function of the interval $(0,1]$, in which case we drop f from the notation. On the other hand, when working with the oscillator representation we typically assume f in $L^2(\mathbb{R})$ or even that f is Schwartz.

1.1 Gauss sums

The historically first theta sums are for rational $\alpha = \frac{a}{q}$, $x = y = 0$, and $N = q$:

$$G(a,q) = \theta(q, 2\frac{a}{q}, 0, 0) = \sum_{n=1}^{q} \mathrm{e}(\frac{an^2}{q}). \tag{2}$$

Due to the complete sum over the period of the exponential, Gauss sums have nice algebraic properties. After Gauss, we can compute their value exactly.

Proposition 1.1. *For a positive integer q, we have*

$$G(1,q) = \begin{cases} (1+\mathrm{i})\sqrt{q} & \text{if } q \equiv 0 \bmod 4 \\ \sqrt{q} & \text{if } q \equiv 1 \bmod 4 \\ 0 & \text{if } q \equiv 2 \bmod 4 \\ \mathrm{i}\sqrt{q} & \text{if } q \equiv 3 \bmod 4. \end{cases} \tag{3}$$

We may see a modern proof of this proposition later. For now, we remark on its importance to number theory. For an odd prime p and integer a, we define the Legendre symbol $(\frac{a}{p})$ by

$$(\frac{a}{p}) = \begin{cases} 0 & \text{if } a = 0 \bmod p \\ 1 & \text{if } x^2 = a \bmod p \text{ has a nonzero solution} \\ -1 & \text{if } x^2 = a \bmod p \text{ does not have a solution.} \end{cases} \tag{4}$$

The connection between the Gauss sums and the Legendre symbol is as follows.

Lemma 1.2. *For a prime p and $a \neq 0 \bmod p$, we have*

$$G(a,p) = (\frac{a}{p})G(1,p). \tag{5}$$

Moreover, for odd primes $p \neq q$, we have

$$G(1,pq) = G(p,q)G(q,p). \tag{6}$$

Proof. Equation (5) follows from the fact the set of nonzero $a \bmod p$ such that $x^2 = a$ has a solution is a subgroup of index 2 in the units mod p. Indeed, either an^2 runs over this subgroup or its coset depending on whether $(\frac{a}{p}) = \pm 1$, and we note that the complete sum of $\mathrm{e}(\frac{n}{p})$ is zero.

Equation (6) follows from the Chinese remainder theorem. Indeed $n = n_1 q + n_2 p$ defines a bijection from the set of $n \bmod pq$ to the set of pairs $(n_1 \bmod p, n_2 \bmod q)$. □

Combining lemma 1.2 with proposition 1.1 leads to the celebrated quadratic reciprocity theorem.

Theorem 1.3. *Let p and q be distinct odd primes. We have*

$$(\frac{p}{q}) = \begin{cases} (\frac{q}{p}) & \text{if } p \text{ or } q = 1 \bmod 4 \\ -(\frac{q}{p}) & \text{if } p \text{ and } q = 3 \bmod 4. \end{cases} \tag{7}$$

Gauss famously gave many 8 rigorous proofs (and several other not-so rigorous proofs) and called it "the fundamental theorem" in his book *Disquisitiones Arithmeticae.* The proof here is one of them; although the proof of proposition 1.1 is the heart of the matter, and our approach will be more modern than Gauss's.

Exercise 1. Prove proposition 1.1 with the following steps (due to Dirichlet):

a) Show that for $q \neq 2 \bmod 4$, $G(1, q^3) = qG(1, q)$.

b) Divide $G(1, q)$ into four sums and approximate by integrals using the Euler-Maclaurin formula or Poisson summation.

c) Compute the integrals and show that any remainder vanishes using part a.

Exercise 2. Using proposition 1.1 and lemma 1.2 to show that for an odd prime p,

$$(\frac{-1}{p}) = \begin{cases} 1 & \text{if } p = 1 \bmod 4 \\ -1 & \text{if } p = -1 \bmod 4 \end{cases} \tag{8}$$

and

$$(\frac{2}{p}) = \begin{cases} 1 & \text{if } p = 1 \bmod 8 \\ -1 & \text{if } p = 3 \bmod 8 \\ -1 & \text{if } p = 5 \bmod 8 \\ 1 & \text{if } p = 7 \bmod 8. \end{cases} \tag{9}$$

1.2 Equidistribution of αn^2

Despite not having the same rich algebraic structure as the complete Gauss sums, the theta sums for irrational α can also be controlled and applied. For example, in the early 20th century, Weyl proved the following.

Theorem 1.4. *For any polynomial P with irrational leading coefficient, the sequence $P(n) \in \mathbb{R}/\mathbb{Z}$ is equidistributed.*

We will prove this theorem for quadratic polynomials. The result for general polynomials follows from similar ideas. Indeed, the equidistribution in general reduces to the quadratic case (linear, ultimately) via the *Weyl differencing* technique, which we use in the proof of theorem 1.7.

Theta sums, and exponential sums in general, which in this context are often called *Weyl sums*, are connected to the question of equidistribution via the *Weyl criterion*.

Proposition 1.5. *The sequence of $x_n \in \mathbb{R}/\mathbb{Z}$ is equidistributed if and only if for every nonzero integer h,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathrm{e}(hx_n) = 0. \tag{10}$$

For quadratic polynomials $\alpha n^2 + ny$, the Weyl sums are examples of theta sums,

$$\theta(N, 2\alpha h, 0, hy) = \sum_{n=1}^{N} \mathrm{e}(\alpha h n^2 + hny). \tag{11}$$

We note that if $\alpha = \frac{a}{q}$ is rational, this theta sum is $\frac{N}{q}$ times a Gauss sum whenever N is a multiple of q. Therefore rational α do not satisfy 10. To prove theorem 1.4, we must therefore be able to quantify how irrational α is. This is done through Dirichlet's theorem on diophantine approximation.

Proposition 1.6. *Let $Q \geq 1$. For any real number α, there exist a positive integer $q \leq Q$ and integer a coprime to q such that*

$$|\alpha - \frac{a}{q}| \leq \frac{1}{qQ}. \tag{12}$$

Moreover, $q = q(Q) \to \infty$ as $Q \to \infty$ if and only if α is irrational.

Exercise 3. Prove proposition 1.6 using the box principle with $\mathbb{R}/\mathbb{Z}$ into Q intervals of length $\frac{1}{Q}$ and the points $q\alpha$ mod 1 with $0 \leq q \leq Q$. Then find an upper bound for the number of $\frac{a}{q}$ that can approximate α to an error $\frac{1}{qQ}$.

Theorem 1.7. *Let $q \leq 2N$ be so that $|2h\alpha - \frac{a}{q}| \leq \frac{1}{2qN}$ be as in proposition 1.6. We have*

$$|\theta(N, 2\alpha h, 0, hy)| \ll Nq^{-\frac{1}{2}} + N^{\frac{1}{2}}(\log q)^{\frac{1}{2}}. \tag{13}$$

In particular, if α is irrational,

$$\lim_{N\to\infty} \frac{1}{N}\theta(N, 2\alpha h, 0, hy) = 0. \tag{14}$$

Proof. Weyl differencing begins by squaring and a change of variables to obtain

$$|\theta(N, 2\alpha h, 0, hy)|^2 = \sum_{1\le m,n\le N} \mathrm{e}(\alpha h(m^2 - n^2) + (m-n)hy) =$$

$$\sum_{|l|<N} \mathrm{e}(\alpha hl^2 + hly) \sum_{1\le n, n+l\le N} \mathrm{e}(2\alpha hln). \quad (15)$$

For a higher degree polynomial, the inner sum over n will be a Weyl sum for a lower degree polynomial, and the process can be repeated. Once a linear polynomial in n is reached, as it is above, we can execute the geometric series using

$$\sum_{k=0}^{K-1} \mathrm{e}(kx) = \frac{\mathrm{e}(Kx) - 1}{\mathrm{e}(x) - 1} \ll \min(K, \frac{1}{||x||}), \quad (16)$$

where $||\cdot||$ is the distance to the nearest integer, to obtain

$$|\theta(N, 2\alpha h, 0, hy)|^2 \ll \sum_{|l|<N} \min(N, \frac{1}{||2\alpha hl||}). \quad (17)$$

Using the approximation $\frac{a}{q}$ to $2\alpha h$ and breaking into progressions modulo q, this bound becomes

$$\frac{N}{q}(N + \sum_{0<|l|\le\frac{1}{2}q} \frac{q}{l}) \ll N^2 q^{-1} + N \log q, \quad (18)$$

giving (13). □

Exercise 4. Prove theorem 1.4 by applying the method of theorem 1.7 inductively on the degree of the polynomial.

The bound (13) is essentially best possible. However, more precise results can be obtained using more information on the diophantine properties of α. We will obtain such bounds using modern methods, but they were known earlier through the Hardy-Littlewood approximate functional equation for theta sums. This functional equation in a sense comes from the functional equation of the Jacobi theta function.

1.3 Jacobi theta function

We now consider the theta sum θ_f with $f(x) = \exp(-\pi x^2)$. The result is a holomorphic function in the parameter $z = \alpha + \mathrm{i}\frac{1}{N^2}$ called the Jacobi theta function,

$$\theta(z) = \theta_f(N, \alpha, 0, 0) = \sum_{n\in\mathbb{Z}} \mathrm{e}(\frac{1}{2}n^2 z). \quad (19)$$

Jacobi actually considered additional theta functions, roughly corresponding to when the shift x and linear phase y in (1) are integers or half integers. These theta functions have a fairly algebraic feel; indeed Jacobi originally used them to put coordinates on elliptic curves. The transformation of these functions under automorphisms of the elliptic curves was therefore of utmost importance.

Proposition 1.8. *We have*

$$\theta(z) = (\frac{i}{z})^{\frac{1}{2}}\theta(-\frac{1}{z}) \tag{20}$$

where the square root is taken to have positive real part on the right half plane.

Proof. This follows almost immediately from the Poisson summation formula,

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \hat{f}(m), \tag{21}$$

where

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)\mathrm{e}(-\xi x)\mathrm{d}x \tag{22}$$

is the (properly normalized) Fourier transform. For $f(x) = \mathrm{e}(\frac{1}{2}x^2 z)$, we have

$$\hat{f}(\xi) = \mathrm{e}(\frac{1}{2z}\xi^2)\int_{\mathbb{R}} \mathrm{e}(\frac{1}{2}z(x-\frac{\xi}{z})^2)\mathrm{d}x = z^{-\frac{1}{2}}\mathrm{e}(\frac{1}{2z}\xi^2)\int_{\mathbb{R}} \mathrm{e}(\frac{1}{2}x^2)\mathrm{d}x. \tag{23}$$

Some care needs to be taken with the change of variables, contour shift, and choice of square root, but (20) follows from this last integral being $\mathrm{e}(\frac{1}{8})$. □

The functional equation (20) together with the periodicity $z \mapsto z+2$, shows the theta function has symmetries under the lattice

$$\Gamma_\theta = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) : ab \equiv cd \equiv 0 \bmod 2\} \subset \mathrm{SL}(2,\mathbb{R}) \tag{24}$$

with the action being by Möbius transformations $z \mapsto \frac{az+b}{cz+d}$ on the hyperbolic plane $\mathbb{H} = \{x + \mathrm{i}y : y > 0\}$. These symmetries involve the Gauss sums considered above.

Theorem 1.9. *For $\gamma \in \Gamma_\theta$ we have*

$$\theta(\gamma z) = \nu(\gamma) j(\gamma, z)^{\frac{1}{2}}\theta(z), \tag{25}$$

where

$$\nu(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = |c|^{-\frac{1}{2}}G(a,c) = \frac{1}{\sqrt{|c|}}\sum_{n \bmod c} \mathrm{e}(\frac{an^2}{c}) \tag{26}$$

and the cocycle j is given by

$$j(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z) = cz + d. \tag{27}$$

Note that here, as above, some care must be taken about the branch of the square root.

Exercise 5. Verify that Γ_θ is a subgroup of index 3 of $\mathrm{SL}(2,\mathbb{Z})$ and show that $\Gamma_\theta\backslash\mathbb{H}$ has two inequivalent cusps, one of width 2 and the other width 1.

Inspired by insights of Ramanujan, Hardy and Littlewood proved an approximate functional equation similar to (20) but for finite theta sums.

Theorem 1.10. *For a positive integer N, $0 < \alpha < 2$, and $0 \le y \le 1$, we have*

$$\theta(N, \alpha, 0, y) = (\frac{i}{\alpha})^{\frac{1}{2}} \theta(\alpha N, -\frac{1}{\alpha}, 0, \frac{y}{\alpha}) + O(\alpha^{-\frac{1}{2}}). \tag{28}$$

This approximate functional equation can be used to estimate the theta sum in terms of the (even) continued fraction expansion of α. Multiplying the sum by $\alpha^{-\frac{1}{2}}$ while shrinking the length of the sum by a factor α is consistent with the expected *square root cancellation*. The error term being large corresponds to strong approximations of α by rationals. Of course keeping track of the error through the inductive continued fraction expansion can be difficult. In what follows we avoid this difficulty, and continued fractions entirely, using dynamics on $\Gamma_\theta \backslash \mathrm{SL}(2, \mathbb{R})$.

2 The oscillator representation

The oscillator representation is a projective representation of $\mathrm{SL}(2, \mathbb{R})$ (in rank one, more generally $\mathrm{Sp}(d, \mathbb{R})$ in rank d) that is defined through intertwining operators for the Schrödinger representations of the Heisenberg group. Theta sums (for $f \in \mathrm{L}^2(\mathbb{R})$) appear very naturally from the intertwining operator for two different models of the oscillator representation.

The connection between between theta sums and the Heisenberg group can be approached by other means. Notably, work by Flaminio and Forni and, in higher rank, Cosentino and Flaminio, study theta sums via renormalization by $SL(2, \mathbb{R})$ on the Heisenberg group. This is closely related to our approach, but one advantage of ours is that we only need to study one representation – the oscillator representation. This leads to quantitatively stronger results in higher rank.

2.1 Heisenberg group

The Heisenberg group H is the set $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ with multiplication given by

$$(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + \tfrac{1}{2}(x_2 y_1 - x_1 y_2)). \tag{29}$$

However, for our construction of the oscillator representation, we prefer a set-up where it's a bit easier to change basis, notationally-speaking. We note that the expression $x_2 y_1 - x_1 y_2$ defines a symplectic form Ω on $\mathbb{R}^2$, i.e. if $\boldsymbol{v}_j = (x_j, y_j)$, then

$$x_2 y_1 - x_1 y_2 = \Omega(\boldsymbol{v}_1, \boldsymbol{v}_2) = \boldsymbol{v_1} J \, {}^t\boldsymbol{v}_2, \text{ where } J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{30}$$

We observe that $\mathrm{SL}(2, \mathbb{R})$ is exactly the group of matrices preserving the symplectic from Ω, i.e. the group of g satisfying $gJ \, {}^t g = J$.

We fix the standard basis $\boldsymbol{q}_0 = (1, 0)$ and $\boldsymbol{p}_0 = (0, 1)$. In physics, $\boldsymbol{q}_0$ stands for position and $\boldsymbol{p}_0$ for momentum. Any basis $\{\boldsymbol{p}, \boldsymbol{q}\}$ for $\mathbb{R}^2$ for which

$$\Omega(x_1 \boldsymbol{q} + y_1 \boldsymbol{p}, x_2 \boldsymbol{q} + y_2 \boldsymbol{p}) = x_2 y_1 - x_1 y_2 \tag{31}$$

is a *symplectic basis*. Obviously $\{\boldsymbol{p}_0, \boldsymbol{q}_0\}$ is a symplectic basis, and any symplectic basis can be obtained from the standard basis via

$$\begin{pmatrix} \boldsymbol{p} \\ \boldsymbol{q} \end{pmatrix} = \begin{pmatrix} \boldsymbol{p}_0 g \\ \boldsymbol{q}_0 g \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \boldsymbol{p}_0 \\ \boldsymbol{q}_0 \end{pmatrix} \tag{32}$$

for some $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$. In other words, any basis with determinant 1 is a symplectic basis, but this does not generalize to rank d, where the relevant group is $\mathrm{Sp}(d, \mathbb{R})$.

Exercise 6. Show that $\mathrm{Sp}(d, \mathbb{R})$, the set of $2d \times 2d$ matrices $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that $gJ\,{}^t g = J$, where $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, is the set of matrices with $A\,{}^t D - B\,{}^t D = I$ and $A\,{}^t B$, $C\,{}^t D$ are symmetric. Show that $g^{-1} = \begin{pmatrix} {}^t D & -{}^t B \\ -{}^t C & {}^t A \end{pmatrix}$.

We now define the Heisenberg group to be the simply-connected Lie group having Lie algebra $\mathbb{R}^2 \times \mathbb{R}\mathfrak{r}$ with bracket given by

$$\begin{aligned} [\boldsymbol{u}, \boldsymbol{v}] &= \Omega(\boldsymbol{u}, \boldsymbol{v})\mathfrak{r} \quad \text{for } \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^2, \\ [\boldsymbol{u} + z\mathfrak{r}, \mathfrak{r}] &= 0 \quad \text{for } \boldsymbol{u} \in \mathbb{R}^2, z \in \mathbb{R}. \end{aligned} \tag{33}$$

Written in terms of a symplectic basis, these are known as the canonical commutation relations. More concretely, we can express the Heisenberg group as the set of exponentials $\exp(\boldsymbol{v} + z\mathfrak{r})$ with multiplication given by

$$\exp(\boldsymbol{v}_1 + z_1\mathfrak{r}) \exp(\boldsymbol{v}_2 + z_2\mathfrak{r}) = \exp\big(\boldsymbol{v}_1 + \boldsymbol{v}_2 + (z_1 + z_2 + \tfrac{1}{2}\Omega(\boldsymbol{v}_1, \boldsymbol{v}_2))\big). \tag{34}$$

Exercise 7. Prove that this is the multiplication corresponding to the bracket (33) by finding 3×3 matrices satisfying (33) and using the matrix exponential

$$\exp(M) = \sum_{k=0}^{\infty} \frac{1}{k!} M^k. \tag{35}$$

2.2 Schrödinger representation

Let $\ell \subset \mathbb{R}^2$ be a line through the origin and consider the corresponding subgroup $L = \exp(\ell + \mathbb{R}\mathfrak{r}) \subset H$. The function $\phi(\exp(\boldsymbol{v} + z\mathfrak{r})) = \mathrm{e}(z)$ on H restricts to a unitary character on L.[1] The Schrödinger representation of H is the induced representation of ϕ from L to H.

We consider the Hilbert space $\mathcal{H}_\ell$ of functions $f : H \to \mathbb{C}$ satisfying

$$f(lh) = \phi(l)^{-1} f(h) \text{ for all } h \in H \text{ and } l \in L \tag{36}$$

[1]It is typical to use the character $\mathrm{e}(\hbar z)$; $\hbar$ referring to reduced Planck's constant. Often people are interested in the *semi-classical limit* when $\hbar \to 0$, but for our purposes we assume $\hbar = 1$.

and

$$\int_{L\backslash H} |f(h)|^2 \mathrm{d}h < \infty. \tag{37}$$

We note that for a symplectic basis $\boldsymbol{q}, \boldsymbol{p}$ and $h = \exp(x\boldsymbol{q} + y\boldsymbol{p} + z\mathfrak{r})$, Haar measure on H is given by $\mathrm{d}h = \mathrm{d}x\mathrm{d}y\mathrm{d}z$. If we have a symplectic basis such that $\ell = \mathbb{R}\boldsymbol{p}$, the integration in (37) can then be taken over $\mathbb{R}\boldsymbol{q}$ with measure $\mathrm{d}x$. As

$$f(\exp(x\boldsymbol{q} + y\boldsymbol{p})) = f(\exp\big(\boldsymbol{p} - \tfrac{1}{2}xy\mathfrak{r}\big)\exp(\boldsymbol{q})) = \mathrm{e}(\tfrac{1}{2}xy)f(\exp(x\boldsymbol{q}), \tag{38}$$

we may identify $\mathcal{H}_\ell$ with $L^2(\mathbb{R})$. We further note that this measure on $L\backslash H$ is unique up to a positive multiple; this makes no difference in (37), but later we will work with operators between the spaces $\mathcal{H}_\ell$ and choose measures so that the operators are unitary.

Now the Schrödinger representation W_ℓ of H on $\mathcal{H}_\ell$ is simply by right translation

$$W_\ell(h)f : h_1 \mapsto f(h_1 h). \tag{39}$$

To see what this representation is doing more concretely, we compute

$$\begin{aligned} W_\ell(h)f(\exp(x_1\boldsymbol{q})) &= f(\exp\big((x_1 + x)\boldsymbol{q} + y\boldsymbol{p} + (z - \tfrac{1}{2}x_1 y)\big)) \\ &= \mathrm{e}(-z + x_1 y + \tfrac{1}{2}xy)f(\exp((x_1 + x)\boldsymbol{q})). \end{aligned} \tag{40}$$

This differs somewhat from the standard Schrödinger representation in physics where momentum $\boldsymbol{p}$ corresponds to translations (generated infinitesimally by $\frac{\mathrm{d}}{\mathrm{d}x}$) and position $\boldsymbol{q}$ to scaling (generated by $2\pi\mathrm{i}x$). The difference can be remedied by considering the physics $\boldsymbol{p}$ and $\boldsymbol{q}$ as being in the dual space, which is then identified with $\mathbb{R}^2$ via the symplectic form Ω.

Exercise 8. Find the representation $\mathrm{d}W_\ell$ of the Heisenberg Lie algebra (by differentiating W_ℓ or otherwise) and compare with the physicists' notion of $\boldsymbol{p}$ corresponding to $\frac{\mathrm{d}}{\mathrm{d}x}$ and $\boldsymbol{p}$ to multiplication by x.

2.3 Oscillator representation abstractly

The Stone-von Neumann theorem implies that the Schrödinger is essentially[2] the only infinite dimensional, irreducible unitary representation of the Heisenberg group, explaining why Heisenberg's matrix mechanics is necessarily equivalent to Schrödinger's wave mechanics.

The Heisenberg group however has a large group of automorphisms: any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$ preserves the symplectic form Ω, so the action on $h = \exp(x\boldsymbol{q} + y\boldsymbol{p} + z\mathfrak{r}) \in H$ given by

$$h^g = \exp(x\boldsymbol{q}g + y\boldsymbol{p}g + z\mathfrak{r}) = \exp((ax + cy)\boldsymbol{q} + (bx + dy)\boldsymbol{p} + z\mathfrak{r}) \tag{41}$$

is an automorphism, and in fact all outer automorphism of H have this form. Now by the Stone- von Neumann theorem, for each $g \in \mathrm{SL}(2,\mathbb{R})$, there must be a unitary operator $R_\ell(g)$ such that

$$R_\ell(g)W_\ell(h^g)R_\ell(g)^{-1} = W_\ell(h). \tag{42}$$

[2]Different choices of Planck's constant give inequivalent representations.

Schur's theorem implies that such an $R_\ell(g)$ is unique up to scalar multiplication, so we have $R_\ell(g_1g_2) = \rho_\ell(g_1, g_2)R_\ell(g_1)R_\ell(g_2)$ for some complex ρ with modulus 1. As such, R_ℓ defines a *projective representation* of $\mathrm{SL}(2,\mathbb{R})$, the oscillator representation.

From associativity in $\mathrm{SL}(2,\mathbb{R})$, ρ_ℓ must satisfy the cocyle equation

$$\rho_\ell(g_1g_2, g_3)\rho_\ell(g_1, g_2) = \rho_\ell(g_1, g_2g_2)\rho_\ell(g_2, g_3). \tag{43}$$

Moreover, changing our definition of $R_\ell(g)$ to $c(g)R_\ell(g)$ for some scalar $c(g)$ changes $\rho_\ell(g_1, g_2)$ to $c(g_1)c(g_2)c(g_1g_2)^{-1}\rho_\ell(g)$. The cocycle ρ_ℓ modulo the trivial cocycles of the form $(g_1, g_2) \mapsto c(g_1g_2)c(g_1)^{-1}c(g_2)^{-1}$ defines an element of the first homology of $\mathrm{SL}(2,\mathbb{R})$. Since $\mathrm{SL}(2,\mathbb{R})$ has a circle has a deformation retract, it has nontrivial cocycles. As it turns out, ρ_ℓ has order two in the homology group.. While not trivial, ρ_ℓ can be trivialized by passing to the connected double cover of $\mathrm{SL}(2,\mathbb{R})$, the metaplectic group $\mathrm{Mp}(2,\mathbb{R})$. The construction of $\mathrm{Mp}(2,\mathbb{R})$ is interesting (it has no faithful, finite dimensional representations) and not too difficult, but it will not be necessary for our purposes.

2.4 Intertwining operators

We now explicitly construct R_ℓ satisfying (42) and find a direct method for computing the corresponding cocycle ρ_ℓ. For $g \in \mathrm{SL}(2,\mathbb{R})$ we let $A(g)$ be the operator on functions f of H given by

$$A(g)f : h \mapsto f(h^g). \tag{44}$$

We observe that the composition $A(g)^{-1}W_\ell(h)A(g)$ maps f to a function sending $h_1 \mapsto f(h_1h^g)$, noting that since $A(g)$ maps $\mathcal{H}_\ell \to \mathcal{H}_{\ell g^{-1}}$, we need $f \in \mathcal{H}_{\ell g}$. Apart from this issue with domains, this is reminiscent of the operator $W_\ell(h^g)$. In fact, up to some technicalities regarding the measure referred to in section 2.2, we have $A(g)^{-1}W_\ell(h)A(g) = W_{\ell g}(h^g)$. In this section we construct intertwining operators $\mathcal{F}_{\ell_1,\ell_2}$ satisfying

$$\mathcal{F}_{\ell_1,\ell_2}W_{\ell_2} = W_{\ell_1}\mathcal{F}_{\ell_1,\ell_2}. \tag{45}$$

The oscillator representation will then be given by $R_\ell(g) = \mathcal{F}_{\ell,\ell g^{-1}}A(g)$.

The operator $\mathcal{F}_{\ell_1,\ell_2}$ is constructed by the method of averaging. In order to get the required transformation (36) under L_1, we simply integrate f over L_1 against the character ϕ,

$$\mathcal{F}_{\ell_1,\ell_2}f : h \mapsto \int_{L_1/(L_1\cap L_2)} f(lh)\phi(l)\mathrm{d}l. \tag{46}$$

We note here that in rank one, we have either equality $L_1 = L_2$ or trivial intersection $L_1 \cap L_2 = \exp(\mathbb{R}\mathfrak{r})$. Unlike in higher rank, the integration over $L_1/(L_1 \cap L_2)$ therefore causes no difficulty: it is either integration over all of L_1 or $\mathcal{F}_{\ell_1,\ell_2}$ is scalar multiplication. Since we would like $R_\ell(g)$ to be unitary, we will later choose the measure here appropriately, depending on g and the measure in in (37). Up to this scaling, we simply take $\mathrm{d}l = \mathrm{d}y$ for $l = \exp(y\boldsymbol{p})$ if ℓ_1 has basis vector $\boldsymbol{p}$.

Clearly $\mathcal{F}_{\ell_1,\ell_2} : \mathcal{H}_{\ell_2} \to \mathcal{H}_{\ell_1}$ and, since it is constructed from left-averaging, it commutes with right-translation. Therefore we have the desired property $W_{\ell_1}\mathcal{F}_{\ell_1,\ell_2} = \mathcal{F}_{\ell_1,\ell_2}W_{\ell_2}$. We also observe that

$$\mathcal{F}_{\ell_1,\ell_2}A(g)f : h \mapsto \int_{L_1/(L_1\cap L_2)} f(l^g h^g)\phi(l)\mathrm{d}l = A(g)\mathcal{F}_{\ell_1 g,\ell_2 g}f(h) \tag{47}$$

by changing variables $l \leftarrow l^{g^{-1}}$. Thus $R_\ell(g)$ indeed intertwines $W_\ell(h)$ and $W_\ell(h^g)$.

For $\ell_1 \neq \ell_2$ we can find a symplectic basis with $\ell_1 = \mathbb{R}\boldsymbol{p}$ and $\ell_2 = \mathbb{R}\boldsymbol{q}$. Now

$$\mathcal{F}_{\ell_1,\ell_2}f : \exp(x\boldsymbol{q}) \mapsto \int_{\mathbb{R}} f(\exp(y\boldsymbol{p})\exp(x\boldsymbol{q}))\mathrm{d}y = \int_{\mathbb{R}} f(\exp(x\boldsymbol{q} + xy\mathfrak{r})\exp(\boldsymbol{p}))\mathrm{d}y. \tag{48}$$

Since $f \in \mathcal{H}_{\ell_2}$ we have

$$\mathcal{F}_{\ell_1,\ell_2}f : h \mapsto \int_{\mathbb{R}} f(\exp(y\boldsymbol{p}))\mathrm{e}(-xy)\mathrm{d}y. \tag{49}$$

Thus $\mathcal{F}_{\ell_1,\ell_2}$ is essentially the Fourier transformation, and from the physics perspective, we see it here interchanging momentum and position. Moreover, we see immediately that $\mathcal{F}_{\ell_1,\ell_2}$ is unitary (with the above choice of coordinates/measures) and has inverse $\mathcal{F}_{\ell_2,\ell_1}$ (again, with these coordinates).

2.5 Maslov index

To compute the cocycle ρ_ℓ, we will need to compare the composition $\mathcal{F}_{\ell_1,\ell_2}\mathcal{F}_{\ell_2,\ell_3}$ and $\mathcal{F}_{\ell_1,\ell_3}$. By Shur's lemma and the irreducibility of W_{ℓ_1}, they are equal up to a scalar multiple. In this section we prove this directly and determine this scalar multiple explicitly. This scalar is closely related to the Maslov index from physics.

We consider only the case when the ℓ_j are distinct. We may therefore pick a symplectic basis so that $\ell_1 = \mathbb{R}\boldsymbol{p}$, $\ell_2 = \mathbb{R}\boldsymbol{q}$, and $\ell_3 = \mathbb{R}(\boldsymbol{q} + \varepsilon\boldsymbol{p})$ where $\varepsilon = \pm 1$. Now for $f \in \mathcal{H}_{\ell_1}$,

$$\begin{aligned}\mathcal{F}_{\ell_1,\ell_2}\mathcal{F}_{\ell_2,\ell_3}\mathcal{F}_{\ell_3,\ell_1} : \exp(x\boldsymbol{q}) &\mapsto \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} f(\exp(w(\boldsymbol{q}+\varepsilon\boldsymbol{p})\exp(u\boldsymbol{q})\exp(v\boldsymbol{p})\exp(x\boldsymbol{q}))\mathrm{d}u\mathrm{d}v\mathrm{d}w \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} f(\exp((u+w+x)\boldsymbol{q}))\mathrm{e}((u+w)v + \tfrac{1}{2}\varepsilon w^2)\mathrm{d}u\mathrm{d}v\mathrm{d}w. \end{aligned} \tag{50}$$

Changing variables $u \leftarrow u - w - x$, we obtain

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f(\exp(u\boldsymbol{q}))\mathrm{e}((u-w)v)\mathrm{d}u\mathrm{d}v \int_{\mathbb{R}} \mathrm{e}(\tfrac{1}{2}\varepsilon w^2)\mathrm{d}w. \tag{51}$$

By Fourier inversion, the first two integrals become simply $f(\exp(x\boldsymbol{q}))$, while the last integral is $\mathrm{e}(\frac{\varepsilon}{8})$. Therefore we have that

$$\mathcal{F}_{\ell_1,\ell_2}\mathcal{F}_{\ell_2,\ell_3} = \mathrm{e}(\tfrac{1}{8}\tau(\ell_1,\ell_2,\ell_3)\mathcal{F}_{\ell_1,\ell_3}, \tag{52}$$

where $\tau(\ell_1,\ell_2,\ell_3) = \varepsilon$ is the *Maslov index* of the three ordered lines ℓ_1, ℓ_2, ℓ_3. We note that for any permutation σ, $\tau(\ell_{\sigma(1)},\ell_{\sigma(2)},\ell_{sigma(3)}) = \mathrm{sign}(\sigma)\tau(\ell_1,\ell_2,\ell_3)$ and direct calculation shows that $\tau(\ell_1, \ell_2, \ell_3) = 0$ if any of the three lines coincide. Moreover, for any $g \in \mathrm{SL}(2, \mathbb{R})$, $\tau(\ell_1 g, \ell_2 g, \ell_3 g) = \tau(\ell_1, \ell_2, \ell_3))$.

The cocycle for the oscillator representation is given by

$$\rho_\ell(g_1, g_2) = \mathrm{e}(\tfrac{1}{8}\tau(\ell, \ell g_1^{-1}, \ell g_2^{-1} g_1^{-1})). \tag{53}$$

The cocycle relation for ρ_ℓ implies that for four lines ℓ_j,

$$\tau(\ell_1, \ell_3, \ell_4) + \tau(\ell_1, \ell_2, \ell_3) = \tau(\ell_1, \ell_2, \ell_4) + \tau(\ell_2, \ell_3, \ell_4), \tag{54}$$

which can be seen geometrically as well.

Exercise 9. Construct the metaplectic group $\mathrm{Mp}(2, \mathbb{R})$ using the Maslov index and show that the oscillator representation is a true representation of $\mathrm{Mp}(2, \mathbb{R})$.

2.6 Oscillator representation explicitly

We now turn towards explicitly computing the projective representation R_ℓ. We choose for the time being $\ell = \mathbb{R}\boldsymbol{p}_0$, and we may drop the subscript ℓ.

We first consider the case when $g = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$. In this case we have $\ell g^{-1} = \ell$ and so $\mathcal{F}_{\ell, \ell g^{-1}}$ is the identity, up to a positive scalar multiple, which we determine by requiring that $R(g)$ is unitary. On the other hand, we have

$$A(g)f : \exp(x\boldsymbol{q}) \mapsto f(\exp x\boldsymbol{q}g) = f(\exp(ax\boldsymbol{q} + bx\boldsymbol{p})) = \mathrm{e}(\tfrac{1}{2}abx^2)f(\exp(ax\boldsymbol{q})). \tag{55}$$

To make $R(g)$ unitary, we must multiply this by $|a|^{\frac{1}{2}}$, so that

$$R(g)f : \exp(x\boldsymbol{q}) = |a|^{\frac{1}{2}}\mathrm{e}(\tfrac{1}{2}abx^2)f(\exp(ax\boldsymbol{q}), \tag{56}$$

or, if we identify $\mathcal{H}_\ell$ with $L^2(\mathbb{R})$, we can write

$$R(g)f(x) = |a|^{\frac{1}{2}}\mathrm{e}(\tfrac{1}{2}abx^2)f(ax). \tag{57}$$

Now we consider $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \neq 0$. As $\ell \neq \ell g^{-1}$, up to a positive scalar multiple, we have that $R(g) = \mathcal{F})\ell, \ell g^{-1} A(g)$ is

$$\begin{aligned} \int_{\mathbb{R}} f(\exp(y\boldsymbol{p}g)\exp(x\boldsymbol{q}g))\mathrm{d}y &= \int_{\mathbb{R}} f(\exp\big((ax+cy)\boldsymbol{q} + (bx+dy)\boldsymbol{p} + \tfrac{1}{2}xy\mathfrak{r}\big))\mathrm{d}y \\ &= \int_{\mathbb{R}} f(\exp((ax+cy)\boldsymbol{q}))\mathrm{e}(\tfrac{1}{2}abx^2 + bcxy + \tfrac{1}{2}cdy^2)\mathrm{d}y \\ &= \frac{1}{c}\int_{\mathbb{R}} f(\exp(y\boldsymbol{q}))\mathrm{e}(\tfrac{1}{2}(\frac{a}{c}x^2 - \frac{2}{c}xy + \frac{d}{c}y^2)\mathrm{d}y. \end{aligned} \tag{58}$$

To find the appropriate scalar fact, one could try to compute directly, but we can also decompose

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \frac{a}{c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & \frac{1}{c} \end{pmatrix}. \tag{59}$$

We note that the first matrix on the write corresponds to multiplication by $\mathrm{e}(\frac{a}{2c}x^2)$ and the second corresponds to Fourier transform. Both are unitary operations, and so we only need to scale by $|c|^{\frac{1}{2}}$ to make $R(g)$ unitary. Therefore, again identifying $\mathcal{H}_\ell$ with $L^2(\mathbb{R})$, we have

$$R(g)f(x) = \frac{\operatorname{sign}(c)}{\sqrt{|c|}} \int_{\mathbb{R}} f(y)\mathrm{e}(\frac{1}{2c}(ax^2 - 2xy + dy^2))\mathrm{d}y. \tag{60}$$

Exercise 10. Show that the function $f(x) = \exp(-\pi x^2)$ is an eigenfunction of all the operators $R(\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix})$ with eigenvalues $\varepsilon\mathrm{e}(\frac{\varepsilon}{8})\mathrm{e}^{-\mathrm{i}\frac{1}{2}\theta}$ when $0 \leq \theta < 2\pi$ and $\varepsilon = \operatorname{sign}(\sin\theta)$. . Find infinitely many other eigenfunctions using Hermite polynomials.

2.7 Weyl quantization

The derivative of the representation R gives a representation $\mathrm{d}R$ of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of $\mathrm{SL}(2,\mathbb{R})$. One complication is that, for our definition, R_ℓ is not continuous at the origin. We therefore compose with the Fourier transform, ie $R_\ell(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})$. Explicitly, we define $\mathrm{d}R$ for $X \in \mathfrak{sl}(2,\mathbb{R})$ by

$$\mathrm{d}R(X)f(h) = \frac{\mathrm{d}}{\mathrm{d}t} R(\exp(tX)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})\hat{f}(h)|_{t=0}. \tag{61}$$

For example, up to scalars, $R(\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix})$ is $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ and $R(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ is multiplication by x^2.

The lie algebra $\mathfrak{sl}(2,\mathbb{R})$ can be identified with quadratic forms $Ax^2 + Bxy + Cy^2$ via

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix} \mapsto \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} B & -A \\ C & -B \end{pmatrix}. \tag{62}$$

Under this identification, up to constants the square of the position coordinate corresponds to the operator $f(x) \mapsto x^2 f(x)$ and the square of momentum corresponds to $f(x) \mapsto f''(x)$. Thus the derivative of the oscillator representation provides a rigorous way to extend the quantization to second order differential operators. This was first constructed by Weyl directly without the oscillator representation.

Exercise 11. Show that $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ corresponds via $\mathrm{d}R$ to a harmonic oscillator. Relate $R(\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix})f(x)$ to solutions of the harmonic oscillator.

3 Theta functions

Theta functions arise from the oscillator representation through a different model of Schrödinger representation. This model has close connections to Floquet theory and to Bloch theory in physics.

3.1 Bloch-Floquet model

We let λ be a lattice generated by a symplectic basis $\boldsymbol{q}, \boldsymbol{p}$ and we set $\Lambda = \exp(\lambda + \mathbb{R}\mathfrak{r})$. We let χ be the character that is trivial on $\mathbb{Z}\boldsymbol{q}$ and $\mathbb{Z}\boldsymbol{p}$ and $\chi(\exp(z\mathfrak{r})) = \mathrm{e}(z)$.

$$\chi(\exp(m\boldsymbol{q} + n\boldsymbol{p})) = (-1)^{mn}. \tag{63}$$

Exercise 12. Show that the stabilizer of χ in $\mathrm{SL}(2,\mathbb{Z})$ is exactly Γ_θ ($h_\gamma = 1$ for $\gamma \in \Gamma_\theta$) and conclude that $\theta_f(1,g)$ is automorphic under Γ_θ as a function of g alone.

We now consider the induced representation W_χ on the Heisenberg group H from χ. Specifically, we let $\mathcal{H}_\chi$ be the Hilbert space of functions f on H satisfying

$$f(uh) = \chi(u)^{-1} f(h) \text{ for all } u \in \Lambda \text{ and } h \in H \tag{64}$$

and

$$\int_{H/\Lambda} |f(h)|^2 \mathrm{d}h < \infty. \tag{65}$$

In contrast to section 2.2, the choice of measure here is natural: for $h = \exp(x\boldsymbol{q} + y\boldsymbol{p})$ we set $\mathrm{d}h = \mathrm{d}x\mathrm{d}y$ so that the total volume of H/Λ is 1.

The representation W_χ is now simply

$$W_\chi(h)f(h_1) = f(h_1 h). \tag{66}$$

By the Stone-von Neumann theorem, there must be an operator $\Theta_{\chi,\ell}$ intertwining W_ℓ and W_χ. We construct such an intertwining operator by averaging assuming that the line ℓ is rational with respect to λ (ie, $\ell \cap \lambda$ is a lattice in λ).

For $f \in \mathcal{H}_\ell$, we define

$$\Theta_{\chi,\ell} f(h) = \sum_{u \in \Lambda/(\Lambda \cap L)} f(uh)\chi(u), \tag{67}$$

and its inverse is given by

$$(\Theta_{\chi,\ell})^{-1} f(h) = \int_{L/(L\cap\Lambda)} f(lh)\phi(l)\mathrm{d}l. \tag{68}$$

Moreover, it is clear that $W_\chi \Theta_{\chi,\ell} = \Theta_{\chi,\ell} W_\ell$.

Exercise 13. Verify that the inverse of $\Theta_{\chi,\ell}$ is as given in (68).

Now for a function $f \in \mathcal{H}_\ell$ and $(h,g) \in H \rtimes G$, we define the theta function by

$$\theta_f(h,g) = \Theta_{\chi,\ell} R_\ell(g) f(h). \tag{69}$$

Let's suppose that $\ell = \mathbb{R}\boldsymbol{p}$ and $h = \exp(x\boldsymbol{q} + y\boldsymbol{p})$, so

$$\theta_f(h,g) = \sum_{m\in\mathbb{Z}} R_\ell(g) f(\exp(m\boldsymbol{q})\exp(x\boldsymbol{q}+y\boldsymbol{p}))$$
$$= \mathrm{e}(\tfrac{1}{2}xy) \sum_{m\in\mathbb{Z}} R_\ell(g) f(\exp((m+x)\boldsymbol{q}))\mathrm{e}(my). \tag{70}$$

Let's further assume that $g = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^{\frac{1}{2}} & 0 \\ 0 & v^{-\frac{1}{2}} \end{pmatrix}$, so that

$$\theta_f(h,g) = v^{\frac{1}{4}} \mathrm{e}(\tfrac{1}{2}xy) \sum_{m\in\mathbb{Z}} f((m+x)v^{\frac{1}{2}}) \mathrm{e}(\tfrac{1}{2}(m+x)^2 u + my). \tag{71}$$

Choosing $v = N^{-2}$ and $u = \alpha$ gives $\mathrm{e}(\frac{1}{2}xy)$ times the theta sum defined at the beginning (1).

3.2 Automorphy of theta functions

The automorphy of the theta functions $\theta_f(h,g)$ defined by (69) is the consequence of the following lemmas.

Lemma 3.1. *For $\gamma \in \mathrm{SL}(2,\mathbb{Z})$, we have*

$$A(\gamma)(\Theta_{\chi,\ell})^{-1} = (\Theta_{\chi^\gamma,\ell\gamma^{-1}})^{-1} A(\gamma), \tag{72}$$

where χ^γ is the function on Λ given by $\chi^\gamma(u) = \chi(u^\gamma)$.

Proof. For $f \in \mathcal{H}_\chi$, we have

$$A(\gamma)(\Theta_{\chi,\ell})^{-1} f(h) = \int_{L/(\Lambda\cap L)} f(lh^\gamma)\phi(l)\mathrm{d}l$$
$$= \int_{L^{\gamma^{-1}}/(\Lambda\cap L^{\gamma^{-1}})} f((lh)^\gamma)\phi(l)\mathrm{d}l = (\Theta_{\chi^\gamma,\ell\gamma^{-1}})^{-1} A(\gamma) f(h), \tag{73}$$

the key point being that if $f \in \mathcal{H}_\chi$, then $A(\gamma)f \in \mathcal{H}_{\chi^\gamma}$. □

If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then we have

$$\chi^\gamma(\exp(m\boldsymbol{q}\gamma + n\boldsymbol{p}\gamma)) = (-1)^{(am+cn)(bm+dn)}. \tag{74}$$

As $(am+cn)(bm+dn) \equiv mn + abm + cdn \bmod 2$, we can express the character $\chi^\gamma(u)$ as $\chi(h_\gamma u h_\gamma^{-1})$, where $h_\gamma = \exp\big(\frac{1}{2}r_\gamma \boldsymbol{q} + \frac{1}{2}s_\gamma \boldsymbol{p}\big)$ with r_γ, s_γ are 0 or 1 depending on if cd, ab are even or odd, respectively.

We denote by $B(h)$ the operator taking f to the function mapping $h_1 \mapsto f(h^{-1}h_1)$.

Lemma 3.2. *We have*

$$\Theta_{\chi,\ell}\mathcal{F}_{\ell,\ell\gamma^{-1}}(\Theta_{\chi^\gamma,\ell\gamma^{-1}})^{-1} = \nu(\gamma)B(h_\gamma) \tag{75}$$

for some scalar $\nu(\gamma)$ *with* $|\nu(\gamma)| = 1$.

Proof. First we recall that

$$\begin{aligned}\mathcal{F}_{\ell,\ell\gamma^{-1}} f(\exp\big(x\boldsymbol{q}\gamma^{-1}\big)) &= |c|^{\frac{1}{2}}\int_{\mathbb{R}} f(\exp(y\boldsymbol{p})\exp\big(x\boldsymbol{q}\gamma^{-1}\big))\mathrm{d}y \\ &= |c|^{\frac{1}{2}}\int_{\mathbb{R}} f(\exp\big((cy+x)\boldsymbol{q}\gamma^{-1}\big))\mathrm{e}(\tfrac{1}{2}cdy^2)\mathrm{d}y \end{aligned}\tag{76}$$

Now we consider the composition

$$\begin{aligned}&(\Theta_{\chi,\ell})^{-1}B_{h_\gamma}\Theta_{\chi,\ell\gamma^{-1}} f(\exp\big(x\boldsymbol{q}\gamma^{-1}\big)) \\ &\quad = \int_0^1 \sum_{m\in\mathbb{Z}} f(\exp\big(m\boldsymbol{q}\gamma^{-1}\big)\exp\big(\tfrac{1}{2}r\boldsymbol{q}\gamma^{-1}+\tfrac{1}{2}s\boldsymbol{p}\gamma^{-1}\big)\exp(y\boldsymbol{p})\exp\big(x\boldsymbol{q}\gamma^{-1}\big))\mathrm{d}y,\end{aligned}\tag{77}$$

where r and s are so that $\exp\big(\frac{1}{2}r\boldsymbol{q}\gamma^{-1}+\frac{1}{2}s\boldsymbol{p}\gamma^{-1}\big) = h_\gamma^{-1}$. We note that $r \equiv ac \bmod 2$ and $s \equiv bc \bmod 2$, or in other words $\exp\big(\frac{1}{2}r\boldsymbol{q}+\frac{1}{2}s\boldsymbol{p}\big) = h_{\gamma^{-1}}$ modulo Λ.

The operator (77) intertwines W_ℓ and $W_{\ell\gamma^{-1}}$ and therefore must be a scalar multiple of $\mathcal{F}_{\ell,\ell g^{-1}}$. We prove this and determine this scalar explicitly by unfolding the integral in (77). As $f \in \mathcal{H}_{\ell\gamma^{-1}}$, the integrand is equal to

$$\mathrm{e}(\tfrac{1}{8}rs+\tfrac{1}{2}ms+\tfrac{1}{2}cdy^2+\tfrac{1}{2}dry+dmy)f(\exp\left(m+\frac{1}{2}r+cy+x\right)\boldsymbol{q}\gamma^{-1}). \tag{78}$$

We first change variables $m \leftarrow cm+n$ with n modulo c and $y \leftarrow y-m$. The integrand becomes

$$\begin{aligned}&\mathrm{e}(\tfrac{1}{8}rs-\tfrac{1}{2}cdm^2+m(\tfrac{1}{2}cs-\tfrac{1}{2}dr-dn)+\tfrac{1}{2}ns+\tfrac{1}{2}cdy^2+dny+\tfrac{1}{2}dry) \\ &\qquad f(\exp\big((cy+n+\tfrac{1}{2}r+x)\boldsymbol{q}\gamma^{-1}\big))\end{aligned}\tag{79}$$

while the range of integration in y becomes $\int_m^{m+1}$. We claim that the dependence on m in the integrand is trivial. Indeed, dmn is an integer and $cdm^2 \equiv (cs-dr)m \bmod 2$ follows from

$$cs-dr \equiv bcd-acd \equiv cd(ad-bc) \bmod 2. \tag{80}$$

We can now unfold the integral to obtain

$$\begin{aligned}&\mathrm{e}(\tfrac{1}{8}rs)\sum_{n \bmod c}\mathrm{e}(\tfrac{1}{2}ns)\int_{\mathbb{R}} f(\exp\big((cy+n+\tfrac{1}{2}r+x)\boldsymbol{q}\gamma^{-1}\big))\mathrm{e}(\tfrac{1}{2}cdy^2+dy(n+\tfrac{1}{2}r))\mathrm{d}y \\ &= \sum_{n \bmod c}\mathrm{e}(\frac{1}{2c}(-dn^2+(cs-dr)n+\tfrac{1}{4}crs-\tfrac{1}{4}dr^2))\int_{\mathbb{R}} f(\exp\big((cy+x)\boldsymbol{q}\gamma^{-1}\big))\mathrm{e}(\frac{1}{2}cdy^2)\mathrm{d}y.\end{aligned}\tag{81}$$

We therefore have (75) with

$$\nu(\gamma)^{-1} = |c|^{-\frac{1}{2}} \sum_{n \bmod c} \mathrm{e}\Big(\frac{1}{2c}(-dn^2 + (cs - dr)n + \tfrac{1}{4}crs - \tfrac{1}{4}dr^2)\Big), \tag{82}$$

noting that $|\hat{(}\gamma)| = 1$ follows from the fact that all the operators are unitary. This also follows from classical results regarding Gauss sums. □

Combining lemmas 3.1 and 3.2, we have that

$$\Theta_{\chi,\ell}\mathcal{F}_{\ell,\ell\gamma^{-1}}A(\gamma) = \nu(\gamma)B(h_\gamma)A(\gamma)\Theta_{\chi,\ell}, \tag{83}$$

and therefore

$$\begin{aligned}\theta_f(h,\gamma g) = \Theta_{\chi,\ell}R_\ell(\gamma g)f(h) &= \nu(\gamma)\rho_\ell(\gamma,g)B(h_\gamma)A(\gamma)\Theta_{\chi,\ell}R_\ell(g)f(h) \\ &= \nu(\gamma)\rho_\ell(\gamma,g)\theta_f((h_\gamma^{-1}h)^\gamma, g).\end{aligned} \tag{84}$$

This is the automorphy of the theta functions. We make it more appealing by replacing $h \leftarrow h_\gamma h^{\gamma^{-1}}$. This corresponds to the multiplication on $H \rtimes G$ given by

$$(h_1,g_1)(h_2,g_2) = (h_1 h_2^{g^{-1}}, g_1 g_2). \tag{85}$$

Theorem 3.3. *We have*

$$\theta_f((h_\gamma,\gamma)(h,g)) = \nu(\gamma)\rho_\ell(\gamma,g)\theta_f(h,g) \tag{86}$$

where

$$\nu(\gamma)^{-1} = |c|^{-\frac{1}{2}} \sum_{n \bmod c} \mathrm{e}\Big(\frac{1}{2c}(-dn^2 + (cs - dr)n + \tfrac{1}{4}crs - \tfrac{1}{4}dr^2)\Big), \tag{87}$$

with r, s either 0 or 1 satisfying $r \equiv ac \bmod 2$ and $s \equiv bd \bmod 2$.

Exercise 14. Use exercise 10 and theorem (86) to prove theorem 1.9.

4 Applications

We now outline some applications of the theory discussed above.

4.1 Bounds for theta sums

In order to obtain pointwise estimates on $theta_f(h,g)$, we cannot work with functions defined only in an L^2 sense. We therefore assume that f is a Schwartz function, meaning there are constants $C(k,A) > 0$ such that

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k} f(x) \le C(k,A)(1+x^2)^{-A} \tag{88}$$

for all $A > 0$ and integers $k \ge 0$.

Lemma 4.1. *There exist constants $C(k,A) > 0$ such that*

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k} R(\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}) f(x) \leq C(k,A)(1+x^2)^{-A} \tag{89}$$

for all $\theta \in [0, 2\pi)$, $A > 0$, and integers $k \geq 0$.

Proof. When $\sin^2\phi \geq \frac{1}{2}$, use integration by parts and the fact that f is Schwartz to obtain the bound. When $\cos^2\theta \geq \frac{1}{2}$, use the same strategy after composing with the Fourier transform. $\square$

Lemma 4.2. *Let f be a Schwartz function and set $f_\theta = R(\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix})f$. We have for $|x| \leq \frac{1}{2}$ and $v > 0$,*

$$\sum_{m \neq 0} |f_\theta((m+x)v^{\frac{1}{2}})| \ll_{f,A} v^{-A} \tag{90}$$

Exercise 15. Prove lemma 4.2.

The following now follows immediately from the above lemmas.

Lemma 4.3. *For a Schwartz function f and $(h,g) \in H \rtimes G$, we have*

$$\theta_f(h,g) = v^{\frac{1}{4}} \mathrm{e}(\tfrac{1}{2}xy + \tfrac{1}{2}ux^2) f_\phi(v^{\frac{1}{2}}x) + O_{f,A}(v^{-A}), \tag{91}$$

where

$$(uh_\gamma h^{\gamma^{-1}}, \gamma g) = (\exp(x\boldsymbol{q} + y\boldsymbol{p}), \begin{pmatrix}1 & u \\ 0 & 1\end{pmatrix}\begin{pmatrix}v^{\frac{1}{2}} & 0 \\ 0 & v^{-\frac{1}{2}}\end{pmatrix}\begin{pmatrix}\cos\phi & -\sin\phi \\ \sin\phi & \cos\phi\end{pmatrix}) \tag{92}$$

with $(uh_\gamma h^{\gamma^{-1}}, \gamma g)$ in the standard fundamental domain.

Theorem 4.4. *Let f be a Schwartz function and $\psi(x) \geq 1$ be an increasing function such that*

$$\sum_{k \geq 0} \psi(k)^{-4} < \infty. \tag{93}$$

There is a set $\mathcal{A}_\psi$ of full measure such that

$$\theta_f(N, \alpha, x, y) \ll_{f,\alpha} N^{\frac{1}{2}} \psi(\log N) \tag{94}$$

for all $\alpha \in \mathcal{A}_\psi$.

Proof. The Haar measure of the set of g in the fundamental domain with height greater than $Y > 1$ is Y^{-1}. By the invariance of the Haar measure, the same is true for the measure of the set of $g\begin{pmatrix}\mathrm{e}^{-t} & 0 \\ 0 & \mathrm{e}^{t}\end{pmatrix}$ with height greater than Y.

Now if for some $t >$ we have that the height is greater than $\psi(t)^4$, then, since the height is log-Lipschitz, there is some integer $k \geq 0$ such that the height is $\gg \psi(k)^4$. It follows that the measure of the set of g such that

$$\theta_f(h, g \begin{pmatrix} \mathrm{e}^{-t} & 0 \\ 0 & \mathrm{e}^t \end{pmatrix}) \geq C\psi(t) \tag{95}$$

for all $C > 0$ is bounded by

$$\lim_{C\to\infty} C^{-1} \sum_{k\geq 0} \psi(k)^4 = 0, \tag{96}$$

assuming the sum converges.

The theorem as stated follows now follows from the unstable-stable decomposition of g into

$$g = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}. \tag{97}$$

Indeed, for $|\log a| \leq \epsilon$ and $|s| < 1$, $g \begin{pmatrix} \mathrm{e}^{-t} & 0 \\ 0 & \mathrm{e}^t \end{pmatrix}$ has large height if and only $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ does as well. $\square$

Exercise 16. Prove the following variations of theorem 5.2.

a) Prove that theorem 5.2 holds for f the indicator function of $(0, 1)$ using a smooth dyadic decomposition

$$f(x) = \sum_{j=0}^{\infty} (f_1(2^j x) + f_2(2^j(1 - x))) \tag{98}$$

and the oscillator representation to interpret the scaling 2^j.

b) Recover theorem 1.7 by showing that the height of $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-t} & 0 \\ 0 & \mathrm{e}^t \end{pmatrix}$ grows like $O(\mathrm{e}^t)$ if and only if α is rational.

c) Prove versions of theorem 5.2 for diophantine α (α satisfying $|\alpha - \frac{a}{q}| \geq \frac{c(\alpha)}{q^\kappa}$ for some $\kappa \geq 2$). For example, prove that $\theta(N, \sqrt{2}, x, y) \ll N^{\frac{1}{2}}$.

d) Show that one can improve the result to $\sum \psi(k)^{-6} < \infty$, but for almost every (α, y) instead of almost every α by using the bound $\theta_f(h, g) \ll_{f,A} v^{\frac{1}{4}}(1 + v^{\frac{1}{2}}|x|)^{-A}$ instead of just $\ll_f v^{\frac{1}{4}}$.

Exercise 17. Prove the Hardy-Littlewood approximate functional equation using (98) to relate the theta sum $theta(N, \alpha, x, y)$ to an infinite sum of $theta_f$ with Schwartz functions f, which can be truncated with reasonable error.

4.2 Computing Gauss sums

One can isolate the Gauss sums from the automorphy (86) by looking at asymptotics of the theta function. Indeed, choosing $f(x) = \mathrm{e}^{-\pi x^2}$ and writing θ as a function of $x + \mathrm{i}y$ via the Iwasawa decomposition, we have $\theta(x + \mathrm{i}y) \sim y^{\frac{1}{4}}$ as $y \to \infty$. If q is odd, we then obtain the classical Gauss sums from the multiplier ν by applying $\gamma = \begin{pmatrix} 2 & b \\ c & 2d \end{pmatrix} \in \Gamma_\theta$, noting that $\gamma\infty = \frac{2}{c}$. The value of the Gauss sum can then be determined by applying $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, which has trivial ν and maps $\frac{2}{q}$ to $-\frac{q}{2}$, and finding the asymptotics of $\theta(-\frac{q}{2} + \mathrm{i}y)$ as $y \to 0$. For this, we note that

$$\sum_{m\in\mathbb{Z}} \mathrm{e}(-\tfrac{1}{4}qm^2 + \tfrac{1}{2}\mathrm{i}m^2 y) = \sum_{n \bmod 4} \mathrm{e}(-\tfrac{1}{4}qn^2) \sum_{m\equiv n \bmod 4} \mathrm{e}^{-m^2 y}. \tag{99}$$

For y small, the sum over m is well approximated by the integral

$$\tfrac{1}{4}\int_{\mathbb{R}} \mathrm{e}^{-\pi u^2 y} \mathrm{d}u = \tfrac{1}{4} y^{-\frac{1}{2}}, \tag{100}$$

while the sum over n only depends on q mod 4.

Exercise 18. Prove proposition 1.1 by filling the details above and finding arguments for when q is even.

5 Higher rank theory

In this section we breifly sketch how this theory extends to higher rank. One of the nice features of our approach is that much of the theory above carries over to theta sums in several areas, but there are key differences. Instead of using lines in $\mathbb{R}^2$ to define the Schrödinger representation, we use lagrangian subspaces of $\mathbb{R}^{2d}$, isotropic subspaces of the symplectic form with dimension d. One of the main complications comes from the fact that distinct subspaces can have nontrivial intersection.

5.1 Heisenberg group and Schrödinger representation

As for rank one, we let Ω denote the symplectic form

$$\Omega(\boldsymbol{v}_1, \boldsymbol{v}_2) = \begin{pmatrix} \boldsymbol{x}_1 & \boldsymbol{y}_1 \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} {}^t\boldsymbol{x}_2 \\ {}^t\boldsymbol{y}_2 \end{pmatrix} = \boldsymbol{y}_1\, {}^t\boldsymbol{x}_2 - \boldsymbol{x}_1\, {}^t\boldsymbol{y}_2. \tag{101}$$

A symplectic basis

$$\mathfrak{Q} = \begin{pmatrix} \boldsymbol{q}_1 \\ \vdots \\ \boldsymbol{q}_d \end{pmatrix}, \quad \mathfrak{P} = \begin{pmatrix} \boldsymbol{p}_1 \\ \vdots \\ \boldsymbol{p}_d \end{pmatrix} \tag{102}$$

is a basis for which $\Omega(\boldsymbol{x}_1\mathfrak{Q}+\boldsymbol{y}_1\mathfrak{P}, \boldsymbol{x}_2\mathfrak{Q}+\boldsymbol{y}_2\mathfrak{P}) = \boldsymbol{y}_1{}^t\boldsymbol{x}_2 - \boldsymbol{x}_1{}^t\boldsymbol{y}_2$, or equivalently, any basis obtained from the standard basis by an element of $\mathrm{Sp}(d,\mathbb{R})$. A lagrangian subspace ℓ of $\mathbb{R}^{2d}$ is a d-dimensional subspace on which $\Omega = 0$, or equivalently, $\ell = \mathbb{R}^d\mathfrak{P}$ for some symplectic basis.

We define the Heisenberg Lie algebra and exponentiate to get the Heisenberg group as the set of exponentials $\exp(\boldsymbol{v}+z\mathfrak{r})$, $\boldsymbol{v}\in\mathbb{R}^d$ and $z\in\mathbb{R}$, with multiplication

$$\exp(\boldsymbol{v}_1+z_1\mathfrak{r})\exp(\boldsymbol{v}_1+z_2\mathfrak{r}) = \exp\big(\boldsymbol{v}_1+\boldsymbol{v}_2+(z_1+z_2+\tfrac{1}{2}\Omega(\boldsymbol{v}_1,\boldsymbol{v}_2\mathfrak{r}))\big). \tag{103}$$

For any lagrangian subspace ℓ, the subgroup $L=\exp(\ell+\mathbb{R}\mathfrak{r})$ is abelian and the function $\phi(\exp(\boldsymbol{v}+z\mathfrak{r}))=\mathrm{e}(z)$ is a unitary character on L.

As in rank one, the Schrödinger representation W_ℓ is induced from ϕ: it acts by right translate $W_\ell(h)f(h_1)=f(h_1h)$ on the Hilbert space $\mathcal{H}_\ell$ of functions f satisfying

$$f(lh)=\phi(l)^{-1}f(h) \text{ for all } l\in L, h\in H, \tag{104}$$

and

$$\int_{H/L}|f(h)|^2\mathrm{d}h<\infty. \tag{105}$$

The Hilbert space H_ℓ can be identified with $L^2(\mathbb{R}^d)$. We compute explicitly

$$\begin{aligned}
&W_\ell(\exp(\boldsymbol{x}\mathfrak{Q}+\boldsymbol{y}\mathfrak{P}+z\mathfrak{r}))f(\exp(\boldsymbol{x}_1\mathfrak{Q}))\\
&\quad= f(\exp\big(\boldsymbol{y}\mathfrak{P}+(z-\tfrac{1}{2}\boldsymbol{x}\,{}^t\boldsymbol{y}-\boldsymbol{x}_1{}^t\boldsymbol{y})\mathfrak{r}\big)\exp((\boldsymbol{x}_1+\boldsymbol{x})\mathfrak{Q}))\\
&\qquad= \mathrm{e}(-z+\tfrac{1}{2}\boldsymbol{x}\,{}^t\boldsymbol{y}+\boldsymbol{x}_1{}^t\boldsymbol{y})f(\exp((\boldsymbol{x}_1+\boldsymbol{x})\mathfrak{Q})).
\end{aligned} \tag{106}$$

5.2 Intertwining operators and the oscillator representation

The intertwining operators $\mathcal{F}_{\ell_1,\ell_2}$ for Lagrangian subspaces ℓ_1, ℓ_2 are defined as before,

$$\mathcal{F}_{\ell_1,\ell_2}f(h)=\int_{L_1/(L_1\cap L_2)}f(lh)\phi(l)\mathrm{d}l. \tag{107}$$

The main complication is that $L_1\cap L_2$ may have any dimension $0,\ldots,d$.

We may find a symplectic basis $\mathfrak{Q}$, $\mathfrak{P}$ so that

$$\mathfrak{Q}=\begin{pmatrix}\mathfrak{Q}^{(1)}\\ \mathfrak{Q}^{(2)}\end{pmatrix},\quad \mathfrak{P}=\begin{pmatrix}\mathfrak{P}^{(1)}\\ \mathfrak{P}^{(2)}\end{pmatrix} \tag{108}$$

us a splitting into d_1 and d_2 rows and where $\ell_1=\mathbb{R}^d\mathfrak{P}$ and $\ell_2=\mathbb{R}^{d_1}\mathfrak{P}^{(1)}+\mathbb{R}^{d_2}\mathfrak{Q}^{(2)}$. We note that this is connected to the $g\in\mathrm{Sp}(d,\mathbb{R})$ such that

$$\begin{pmatrix}\mathfrak{Q}^{(1)}g\\ \mathfrak{Q}^{(2)}g\\ \mathfrak{P}^{(1)}g\\ \mathfrak{P}^{(2)}g\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&0&0&-1\\0&0&1&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}\mathfrak{Q}^{(1)}\\ \mathfrak{Q}^{(2)}\\ \mathfrak{P}^{(1)}\\ \mathfrak{P}^{(2)}\end{pmatrix}. \tag{109}$$

For the integration, we may replace $L_1/(L_1 \cap L_2)$ with $\mathbb{R}^{d_2}\mathfrak{P}^{(2)}$, ie

$$\mathcal{F}_{\ell_1,\ell_2} f(\exp(\boldsymbol{x}\mathfrak{Q})) = \int_{\mathbb{R}^{d_2}} f(\exp\big(\boldsymbol{y}\mathfrak{P}^{(2)}\big)\exp\big(\boldsymbol{x}^{(1)}\mathfrak{Q}^{(1)} + \boldsymbol{x}^{(2)}\mathfrak{Q}^{(2)}\big))\mathrm{d}\boldsymbol{y}$$
$$= \int_{\mathbb{R}^{d_2}} f(\exp\big(\boldsymbol{x}^{(1)} + \boldsymbol{y}\mathfrak{P}^{(2)}\big))\mathrm{e}(-\boldsymbol{x}^{(2)}\,{}^t\boldsymbol{y})\mathrm{d}\boldsymbol{y}. \quad (110)$$

We see therefore is a partial Fourier transform. As before, this implies that $\mathcal{F}$ is unitary and $\mathcal{F}^{-1}_{\ell_1,\ell_2} = \mathcal{F}_{\ell_2,\ell_1}$.

The oscillator representation is then given by $R_\ell(g) = \mathcal{F}_{\ell,\ell g^{-1}} A(g)$. As before R_ℓ satisfies the abstract definition; it intertwines the representations $W_{\mathrm{e}}(h)$ and $W_\ell(h^g)$.

We can express $R_\ell(g)$ explicitly using a basis related to (109). Indeed, the Bruhat decomposition with respect to the parabolic subgroup $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ with $n \times n$ blocks is

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ 0 & {}^tA_1^{-1} \end{pmatrix} \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \end{pmatrix} \begin{pmatrix} A_2 & B_2 \\ 0 & {}^tA_2^{-1} \end{pmatrix}, \quad (111)$$

The dimension d_2 of the splitting is equal to the rank of C, and we caution that the A_j, B_j are not unique. We have

$$R_\ell(g) f(\exp(\boldsymbol{x}\mathfrak{Q})) = \mathrm{e}(\tfrac{1}{2}\boldsymbol{x}A_1\,{}^tB_1\,{}^t\boldsymbol{x}) \int_{\mathbb{R}^{d_2}} f(\exp(y\boldsymbol{A}_2\mathfrak{Q}))\mathrm{e}(\tfrac{1}{2}\boldsymbol{y}A_2\,{}^tB_2\,{}^t\boldsymbol{y} - \boldsymbol{x}A_1\,{}^t\boldsymbol{y})\mathrm{d}\boldsymbol{y}. \quad (112)$$

Exercise 19. Show that $f(\boldsymbol{x}) = \exp(= \pi\boldsymbol{x}\,{}^t\boldsymbol{x})$ is an eigenfunction of all the $R_\ell(\begin{pmatrix} C & -S \\ S & C \end{pmatrix})$, where $C + \mathrm{i}S \in \mathrm{U}(d)$.

5.3 Theta functions

Given a symplectic lattice $\lambda = \mathbb{Z}^d\mathfrak{Q} + \mathbb{Z}^d\mathfrak{P}$ set $\Lambda = \exp(\lambda + \mathbb{R}z)$ and let χ a character on Λ that is trivial on $\mathbb{Z}^d\mathfrak{Q}$ and $\mathbb{Z}^d mfP$ and $\chi(\exp(z\mathfrak{r})) = \mathrm{e}(z)$, ie

$$\chi(\exp(\boldsymbol{m}\mathfrak{Q} + \boldsymbol{n}\mathfrak{P}) = \chi(\exp(\boldsymbol{m}\mathfrak{Q})\exp(\boldsymbol{n}\mathfrak{P})\exp\big(\tfrac{1}{2}\boldsymbol{m}\,{}^t\boldsymbol{n}\mathfrak{r}\big)) = (-1)^{\boldsymbol{m}\,{}^t\boldsymbol{n}}. \quad (113)$$

Let W_χ be the induced representation of χ from Λ to H. This means $W_\chi(h)$ acts on the Hilbert space $\mathcal{H}_\chi$ of functions satisfying

$$f(uh) = \chi(u)^{-1} f(h) \text{ for all } u \in \Lambda, h \in H \quad (114)$$

and

$$\int_{\Lambda\backslash H} |f(h)|^2 \mathrm{d}h \quad (115)$$

by translation $W_\chi(h)f(h_1) = f(h_1 h)$. We note that the measure in (115) is $\mathrm{d}h = \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}$ for $h = \exp(\boldsymbol{x}\mathfrak{Q} + \boldsymbol{y}\mathfrak{P})$, so in particular the volume of $\Lambda\backslash H$ is 1.

The operator $\Theta_{\chi,\ell} : \mathcal{H}_\ell \to \mathcal{H}_\chi$ given by

$$\Theta_{\chi,\ell} f(h) = \sum_{u\in(\Lambda\cap L)\backslash\Lambda} f(uh) \tag{116}$$

has inverse

$$(\Theta_{\chi,\ell})^{-1} f(h) = \int_{L/(L\cap\Lambda)} f(lh)\phi(l)\mathrm{d}l \tag{117}$$

and intertwines the representations W_ℓ and W_χ. For $f \in \mathcal{H}_\ell$, we define the theta function on $H \rtimes G$ by

$$\theta_f(h,g) = \Theta_{\chi,\ell} R_\ell(g) f(h). \tag{118}$$

If for $U = C + \mathrm{i}S \in \mathrm{U}(d)$, we set $f_U = R_\ell(\begin{pmatrix} C & -S \\ S & C \end{pmatrix})$, then for $g = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}\begin{pmatrix} Y^{\frac{1}{2}} & 0 \\ 0 & {}^tY^{-\frac{1}{2}} \end{pmatrix}\begin{pmatrix} C & -S \\ S & C \end{pmatrix}$ and $h = (\boldsymbol{x}, \boldsymbol{y}, z)$, we have

$$\theta_f(h,g) = \mathrm{e}(-z + \tfrac{1}{2}\boldsymbol{x}\,{}^t\boldsymbol{y}) \sum_{\boldsymbol{m}} f((\boldsymbol{m}+\boldsymbol{x})Y^{\frac{1}{2}})\mathrm{e}(\tfrac{1}{2}(\boldsymbol{x}+\boldsymbol{m})X\,{}^t(\boldsymbol{x}+\boldsymbol{m}) + \boldsymbol{m}\,{}^t\boldsymbol{y}) \tag{119}$$

The automorphy of the theta function follows as above by a calculation

$$(\Theta_{\chi,\ell})^{-1} B(h_\gamma)\Theta_{\chi^\gamma,\ell\gamma^{-1}} = \nu(\gamma)^{-1}\mathcal{F}_{\ell,\ell\gamma^{-1}} \tag{120}$$

with $\nu(\gamma)^{-1}$ a Gauss sum.

5.4 Bounds in higher rank

Using the same technique as before, we can prove the following theorem.

Theorem 5.1. *Let ψ be an increasing function such that*

$$\sum_{k\geq 0} \psi(k)^{-2d-2} < \infty. \tag{121}$$

Then there exists a subset $\mathcal{Q}_\psi$ of quadratic forms with full measure such that for all $Q \in \mathcal{Q}_\psi$ we have for any Schwartz function f

$$\sum_{\boldsymbol{m}\in\mathbb{Z}^d} f(\frac{1}{N}(\boldsymbol{m}+\boldsymbol{x}))\mathrm{e}(Q(\boldsymbol{m}+\boldsymbol{x}) + \boldsymbol{m}\,{}^t\boldsymbol{y}) \ll_{f,Q} N^{\frac{d}{2}}\psi(\log N). \tag{122}$$

Similarly, we also have the following.

Theorem 5.2. *Let ψ be an increasing function such that*

$$\sum_{k\geq 0}\psi(k)^{-2d-4}<\infty. \tag{123}$$

Then there exists a subset of full measure $\tilde{\mathcal{Q}}_\psi$ of pairs $(Q,\boldsymbol{b})$, Q a quadratic forms and $\boldsymbol{y}\in\mathbb{R}^d$ such that for all $(Q,\boldsymbol{y})\in\mathcal{Q}_\psi$ we have for any Schwartz function f

$$\sum_{\boldsymbol{m}\in\mathbb{Z}^d} f(\frac{1}{N}(\boldsymbol{m}+\boldsymbol{x}))\mathrm{e}(Q(\boldsymbol{m}+\boldsymbol{x})+\boldsymbol{m}^{\,t}\boldsymbol{y})\ll_{f,Q,\boldsymbol{y}} N^{\frac{d}{2}}\psi(\log N). \tag{124}$$

The analogue of lemma 4.1 is straightforward by decomposing

$$\begin{pmatrix} C & -S\\ S & C\end{pmatrix}=\begin{pmatrix} Q_1 & 0\\ 0 & Q_1\end{pmatrix}\begin{pmatrix} C_1 & -S_1\\ S_1 & C_1\end{pmatrix}\begin{pmatrix} Q_2 & 0\\ 0 & Q_2\end{pmatrix} \tag{125}$$

with $Q_1,Q_2\in\mathrm{O}(d)$ and C_1, S_1 diagonal with entries $\cos\theta_j$, $\sin\theta_j$. Lemma 4.2 also generalizes straightforwardly with an appropriate choice of fundamental domain (need basis vectors of the lattice $\mathbb{Z}^dY^{\frac{1}{2}}$ to have angles bounded away from 0). We note that for theorem (124) it was also necessary to have a fundamental with "box-shaped cusps," meaning that high in the cusp, the fundamental domain becomes a product of subsets corresponding the Langlands decomposition of $\mathrm{Sp}(d,\mathbb{R})$.

Once can also prove versions of these theorems with f an indicator of $(0,1]$ using the smooth dyadic decomposition. A small inconvenience is that in this case, we are not sure if one can replace the sum over $\mathbb{Z}^d$ with an irrational lattice.