#1. Equidistribution of long closed horocycles on hyperbolic surfaces

Andreas Strömbergsson
Uppsala University
http://www.math.uu.se/∼astrombe

Homogenous Dynamical systems

Let G – a connected Lie group

μ – a left Haar measure on G

Γ – a discrete subgroup of G.

Set $\boxed{X = \Gamma\backslash G = \{\Gamma g \;:\; g \in G\}}$ a *homogeneous space.*

G acts on X from the right: $(\Gamma g) \cdot g' := \Gamma(gg') \qquad (g, g' \in G)$.

μ descends to a Borel measure on X *which we also call* μ.

Assume $\mu(X) < \infty \overset{\text{Def}}{\Leftrightarrow} \Gamma$ is a *lattice* in G. Then μ on G is also *right* G-invariant; hence μ on X is G-invariant. We normalize μ so that $\mu(X) = 1$.

Let $(h_t)_{t\in\mathbb{R}}$ be a *1-parameter subgroup of* G. (That is, the map $t \mapsto h_t$ is a Lie group homomorphism from $\mathbb{R}$ to G.)

This $(h_t)_{t\in\mathbb{R}}$ gives rise to a *("homogeneous") flow* $(\Phi_t)_{t\in\mathbb{R}}$ on X: $\boxed{\Phi_t(x) := xh_t}$

Note that Φ_t *preserves* μ.

(X, Φ_t) is called a *homogeneous dynamical system.*

Theorem (Ratner, 1991): If $\{h_t\}$ is *Ad-unipotent* then (I) every ergodic Φ_t-invariant probability measure on X is *homogeneous,* and (II) every Φ_t-orbit closure is *homogeneous*, and the orbit *equidistributes* in its closure.

Part (II) in detail: Given any $x \in X$, there exists a closed connected Lie subgroup $H < G$ such that $\{h_t\} \subset H$ and $\overline{\{\Phi_t(x) \ : \ t \in \mathbb{R}\}} = xH$, and this xH is a closed regular submanifold of X which possesses a unique H-invariant probability measure ν_x. Furthermore (equidistribution): For any $f \in C_b(xH)$,

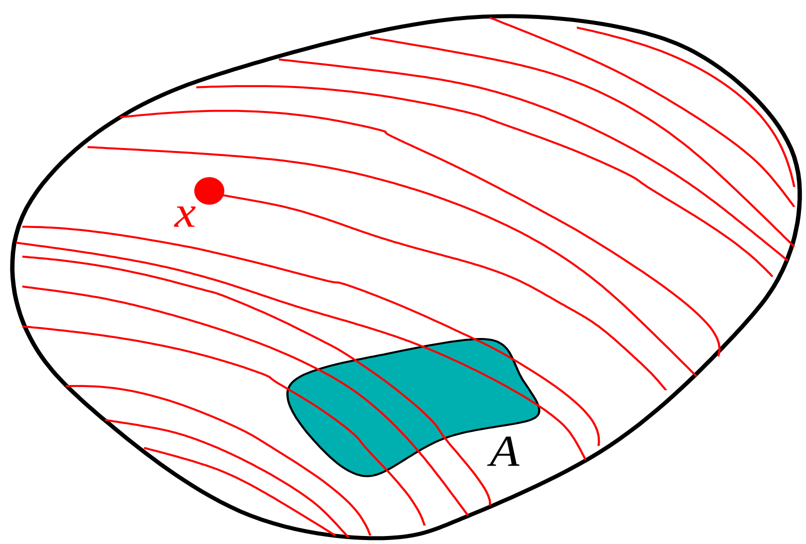
$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(\Phi_t(x))\, dt = \int_{xH} f \, d\nu_x.$$

Theorem (Ratner, 1991): Part (II) in detail: Given any $x \in X$, there exists a closed connected Lie subgroup $H < G$ such that $\{h_t\} \subset H$ and $\overline{\{\Phi_t(x) : t \in \mathbb{R}\}} = xH$, and this xH is a closed regular submanifold of X which possesses a unique H-invariant probability measure ν_x. Furthermore (equidistribution): For any $f \in C_b(xH)$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(\Phi_t(x))\,dt = \int_{xH} f\,d\nu_x.$$

Equidistribution statement $\Leftrightarrow$ For any $A \subset xH$ with $\nu_x(\partial A) = 0$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \chi_A(\Phi_t(x))\,dt = \nu_x(A).$$

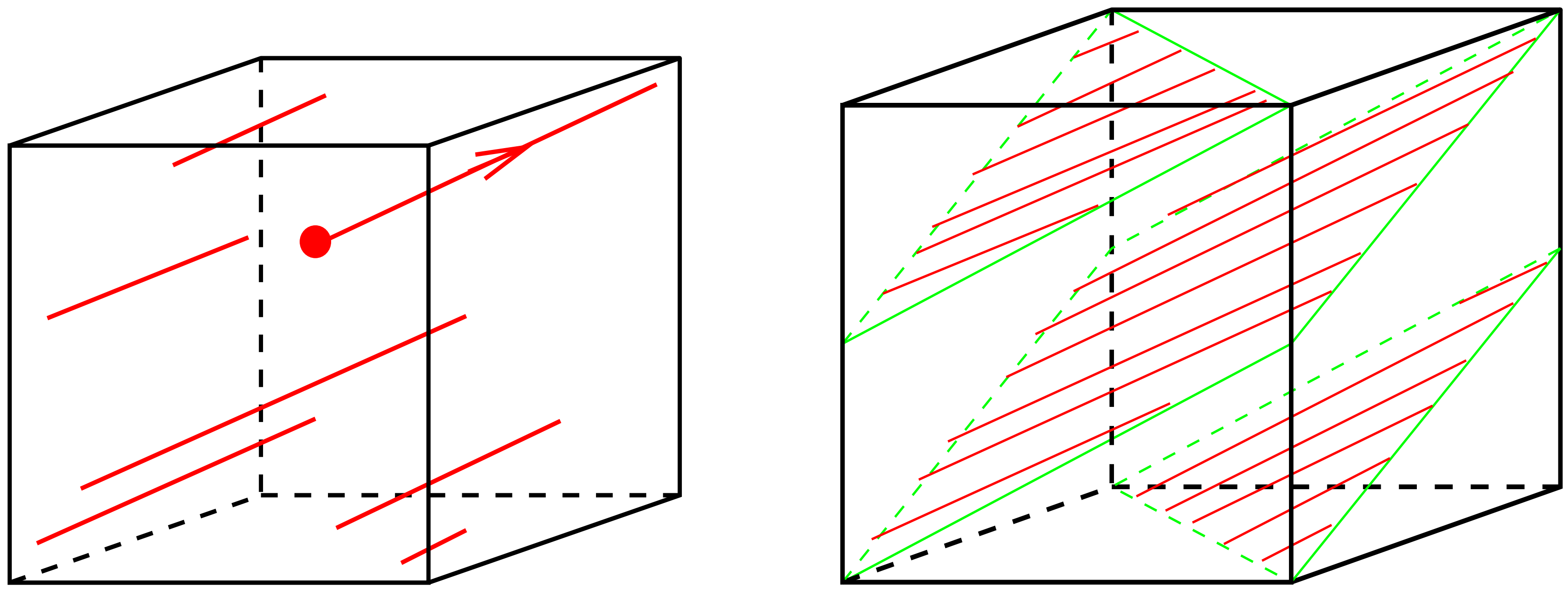


Ratner's Theorem; "trivial" example (Weyl equidistribution)

$G = \mathbb{R}^d$, $\Gamma = \mathbb{Z}^d$; thus $X = \Gamma \backslash G$ a *torus.* μ =Leb.

$h_t = t\,\vec{v}$ for some fixed $\vec{v} \in \mathbb{R}^d$; this gives *linear flow* on the torus X.

Then Ratner's Theorem applies, and "H" is always a *rational linear subspace of* $\mathbb{R}^d$ (which only depends on (h_t), not on x).



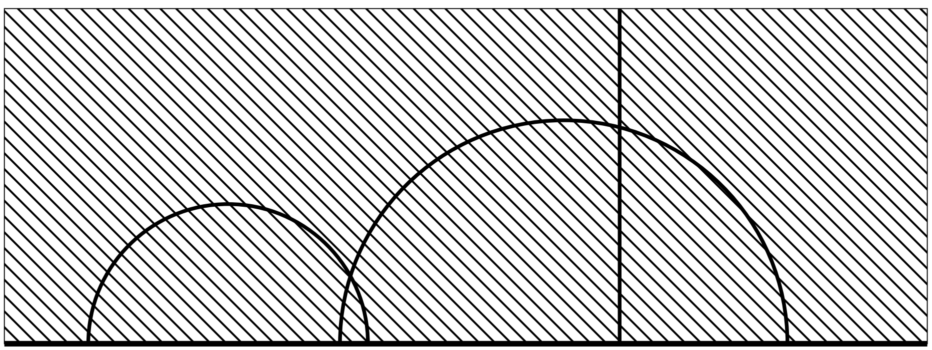
Now let $\boxed{G = \mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm I_2\}}$

Let $\mathbb{H} := \{z = x + iy \in \mathbb{C} : y > 0\}$, with the Riemannian metric $\dfrac{dx^2 + dy^2}{y^2}$.

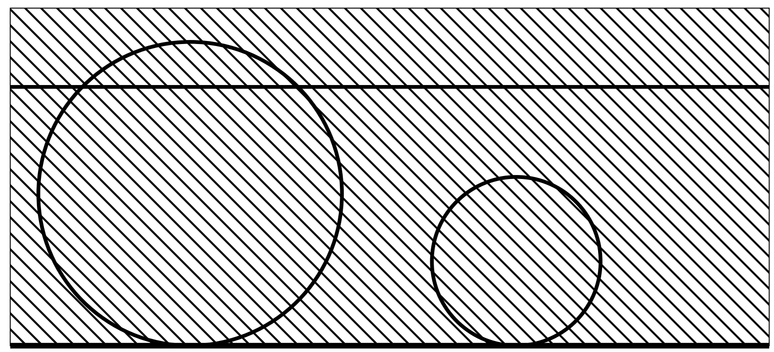
– the Poincaré upper half plane model of the hyperbolic plane.

Area: $\dfrac{dx\,dy}{y^2}$. Length of curve $c : [0,1] \to \mathbb{H}$: $\displaystyle\int_0^1 \frac{|c'(t)|}{\operatorname{Im} c(t)}\,dt$.

Geodesics:



Horocycles:



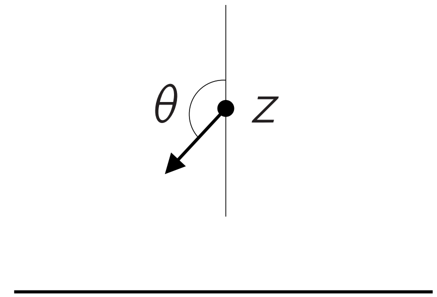
$G = \mathrm{PSL}(2,\mathbb{R})$ acts by orientation preserving isometries on $\mathbb{H}$:

For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{R})$, $z \in \mathbb{H}$: $\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) := \dfrac{az+b}{cz+d}.}$

Identification with $\mathrm{T}^1\mathbb{H}$

Let $\mathrm{T}^1\mathbb{H} := \{v \in \mathrm{T}\mathbb{H} : |v| = 1\}$, the *unit tangent bundle* of $\mathbb{H}$. Parametrization:

$$T^1\mathbb{H} = \big\{(z, \theta) \in \mathbb{H} \times (\mathbb{R}/2\pi\mathbb{Z})\big\}$$



The action $G \times \mathbb{H} \to \mathbb{H}$ has a natural extension to an action $G \times \mathrm{T}^1\mathbb{H} \to \mathrm{T}^1\mathbb{H}$, given by:

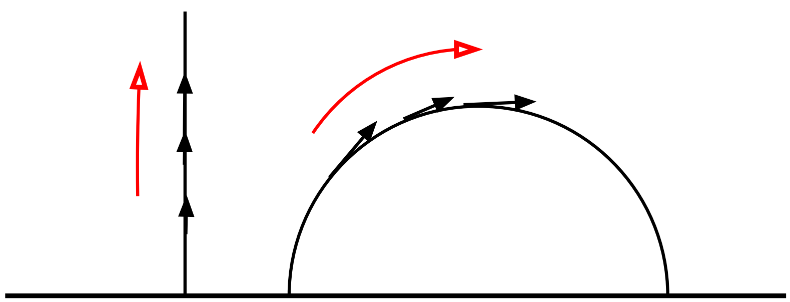
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z, \theta) = \left(\frac{az + b}{cz + d}, \;\; \theta - 2\arg(cz + d) \right).$$

This action is free and transitive; hence for any fixed $p_0 \in \mathrm{T}^1\mathbb{H}$ we have a diffeomorphism $\boxed{G \xrightarrow{\approx} \mathrm{T}^1\mathbb{H}, \quad g \mapsto gp_0}$ Standard choice: $p_0 = (i, 0)$.

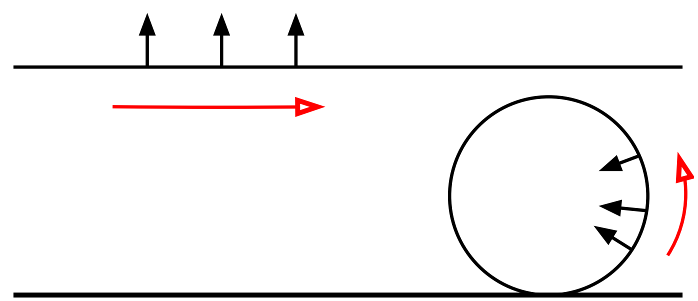
Identifying $G = \mathrm{PSL}(2,\mathbb{R})$ with $\mathrm{T}^1\mathbb{H}$ through $\boxed{G \xrightarrow{\approx} \mathrm{T}^1\mathbb{H}, \quad g \mapsto gp_0}$,

the flow $\boxed{\Phi_t(g) = g\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}}$ on G gives **geodesic flow** on $\mathrm{T}^1\mathbb{H}$,

and the flow $\boxed{\Phi_t(g) = g\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}}$ on G gives **horocycle flow** on $\mathrm{T}^1\mathbb{H}$.



Geodesic flow

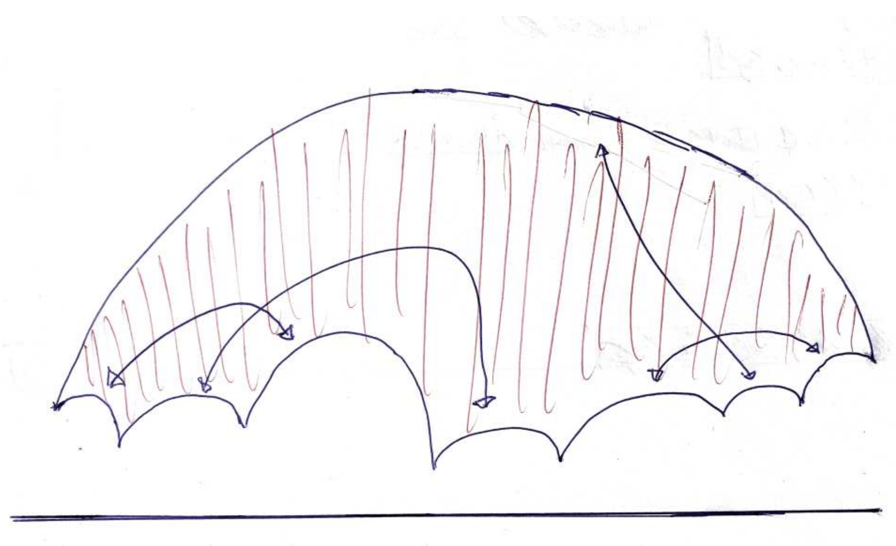


Horocycle flow

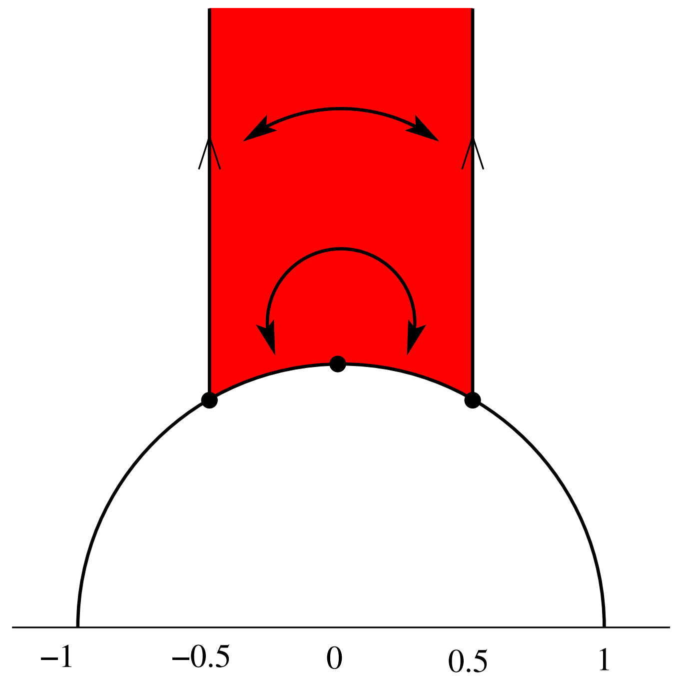
Now let Γ be a discrete subgroup of $G = \mathrm{PSL}(2,\mathbb{R})$

Set $\boxed{M := \Gamma\backslash\mathbb{H}}$, that is, $\mathbb{H}$ with z, z' identified iff $[\exists \gamma \in \Gamma \text{ s.t. } \gamma(z) = z']$. This is a 2-dim *hyperbolic surface, possibly with some cone singularities* (such occur iff Γ contains elliptic elements).

Γ is a lattice in G iff $\mathrm{Area}(M) < \infty$. Then one can find a *fundamental domain* $F \subset \mathbb{H}$ for $\Gamma\backslash\mathbb{H}$ bounded by a finite number of geodesic sides.



Ex 1.



Ex: $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$.

Using $G = \mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{T}^1\mathbb{H}$ we get

$$X = \Gamma\backslash G \;\cong\; \Gamma\backslash \mathrm{T}^1\mathbb{H} = \mathrm{T}^1 M$$

(at least if Γ contains no elliptics).

μ on X gives the *Liouville measure* on $\mathrm{T}^1 M$ (scaled).

The flow $\boxed{\Phi_t(x) = x \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}}$ on X is **geodesic flow** on $\mathrm{T}^1 M$;

the flow $\boxed{\Phi_t(x) = x \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}}$ on X is **horocycle flow** on $\mathrm{T}^1 M$.

– These two flows have *very different* properties!

– The horocycle flow is (Ad-)unipotent; hence Ratner's Theorem applies.
In fact, *every non*-closed Φ_t-orbit *equidistributes* in $\Gamma\backslash G$ (Dani & Smillie, 84).

For the **horocycle flow** on X $\left(\text{i.e., } \Phi_t(x) = x \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right)$; **closed orbits?**

If $\Phi_s(x) = x$ for some $s > 0$, and $x = \Gamma g$ $(g \in G)$, then

$$\Gamma g \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} = \Gamma g, \qquad \text{that is,} \qquad g \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} g^{-1} \in \Gamma.$$

This means that $\Gamma \backslash \mathbb{H}$ has a *cusp* at the point $\eta := g(\infty) \in \partial \mathbb{H}$.
($\Rightarrow$ $\Gamma \backslash \mathbb{H}$ non-compact!)

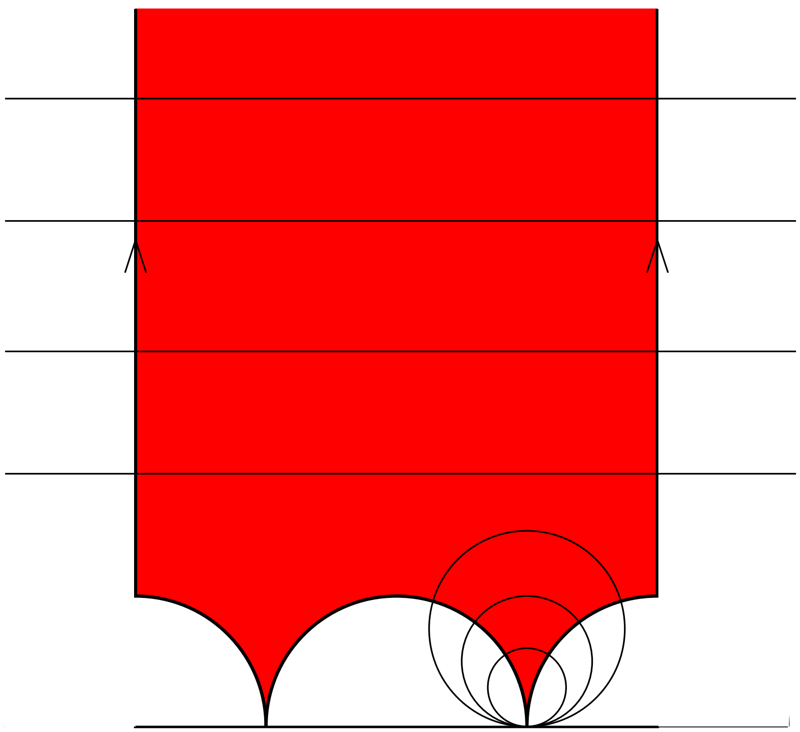
$\left(\text{Here } \partial\mathbb{H} = \mathbb{R} \cup \{\infty\}, \text{ and } G \text{ acts on } \partial\mathbb{H} \text{ by } \begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) = \frac{az+b}{cz+d}. \right)$

Also, *every* g' with $g'(\infty) = \eta = g(\infty)$ is of the form

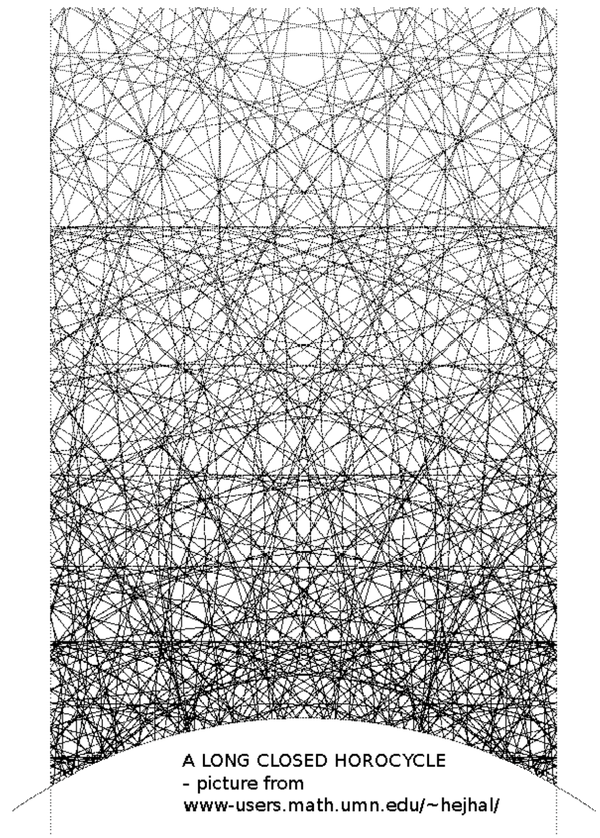
$$g' = g \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \qquad (a \in \mathbb{R}_{>0}, \ x \in \mathbb{R}).$$

Thus we get a *1-parameter family of closed horocycles* associated to η.

Ex: $\Gamma\backslash\mathbb{H}$ **with** 3 **cusps**



Ex: $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$**, a long closed horocycle on** $\Gamma\backslash\mathbb{H}$



Equidistribution of (pieces of) long closed horocycles

Theorem (Selberg; Zagier 1979; Sarnak 1981): Let Γ be a (non-cocompact) lattice in $G = \mathrm{PSL}(2,\mathbb{R})$, let η be a cusp of $\Gamma\backslash\mathbb{H}$, and let $\{H_\ell : \ell \in \mathbb{R}_{>0}\}$ be the associated 1-parameter family of closed horocycles on $X = \Gamma\backslash G$, parametrized so that H_ℓ has length ℓ. Then H_ℓ becomes asymptotically equidistributed in $X = \Gamma\backslash G$ as $\ell \to \infty$, viz., if ν_ℓ is the unit normalized length measure along H_ℓ, then for every $f \in C_b(X)$,

$$\lim_{\ell\to\infty} \int_{H_\ell} f \, d\nu_\ell = \int_X f \, d\mu.$$

(S, '04): In fact, for any $\delta > 0$, if H'_ℓ is a subsegment of H_ℓ of length $\geq \ell^{\frac{1}{2}+\delta}$, then also H'_ℓ become asymptotically equidistributed in $X = \Gamma\backslash G$ as $\ell \to \infty$.

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(S, '04): In fact, for any $\delta > 0$, if H'_ℓ is a subsegment of H_ℓ of length $\geq \ell^{\frac{1}{2}+\delta}$, then also H'_ℓ become asymptotically equidistributed in $X = \Gamma\backslash G$ as $\ell \to \infty$.

Zagier 1979: For $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$, $\int_{H_\ell} f\, d\nu_\ell = \int_X f\, d\mu + O_{f,\varepsilon}\big(\ell^{-\frac{3}{4}+\varepsilon}\big)$ as $\ell \to +\infty$ for every $f \in C_c^\infty(M)$ iff the Riemann Hypothesis holds!

Equidistribution of pieces of long closed horocycles – error term

After a conjugation we may assume that $\eta = \infty$ and $\Gamma_\infty = \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right) \rangle$.

Theorem (S, '13): Let Γ be a lattice in $G = \mathrm{PSL}(2, \mathbb{R})$ such that ∞ is a cusp of $\Gamma \backslash \mathbb{H}$ and $\Gamma_\infty = \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right) \rangle$.

If there exist small eigenvalues $0 < \lambda < \frac{1}{4}$ of the Laplace operator on $\Gamma \backslash \mathbb{H}$, let λ_1 be the smallest of these and define $\frac{1}{2} < s_1 < 1$ so that $\lambda_1 = s_1(1 - s_1)$; otherwise let $s_1 = \frac{1}{2}$.

Similarly define $\frac{1}{2} \le s_1' \le s_1$ from the smallest *non-cuspidal* eigenvalue.

Let $f \in C^3(X)$ with $\|f\|_{W_3} < \infty$, and let $0 < y \le 1$ and $\alpha < \beta \le \alpha + 1$. Then:

$$\frac{1}{\beta - \alpha} \int_\alpha^\beta f(x + iy, 0)\, dx = \int_X f \, d\mu$$

$$+ O\Big(\|f\|_{W_3}\Big) \cdot \left\{ \frac{\sqrt{y}}{\beta - \alpha} \Big(\log\big(1 + y^{-1}\big) \Big)^2 + \Big(\frac{\sqrt{y}}{\beta - \alpha} \Big)^{2(1 - s_1')} + \Big(\frac{y}{\beta - \alpha} \Big)^{1 - s_1} \right\}.$$

– Proof in *next* lecture! Today: How prove such a result *on M* (not X)!?

#2. Equidistribution of long horocycles in $\Gamma\backslash\operatorname{PSL}(2,\mathbb{R})$

Andreas Strömbergsson
Uppsala University
http://www.math.uu.se/∼astrombe

Theorem about long (finite) horocycle orbits (S' 2013)

Let $G = \mathrm{PSL}(2, \mathbb{R})$, let $\Gamma < G$ be a lattice, and set $X := \Gamma \backslash G$.

Recall "$X = \mathrm{T}^1 M$", the unit tangent bundle of the hyperbolic surface $M := \Gamma \backslash \mathbb{H}$.
Projection map $\pi : X \to M$; $\quad \pi(\Gamma g) = \Gamma g(i)$.

Let $n(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$; then $\Phi_t(p) := p\, n(t)$ is the *horocycle flow* on $X = \mathrm{T}^1 M$.

Let $\Delta = -y^2 \big(\partial_x^2 + \partial_y^2 \big)$, the Laplace-Beltrami operator on $\mathbb{H}$ and on $M = \Gamma \backslash \mathbb{H}$.

If there exist small eigenvalues $0 < \lambda < \frac{1}{4}$ of Δ on $M = \Gamma \backslash \mathbb{H}$, let λ_1 be the smallest of these and define $\frac{1}{2} < s_1 < 1$ so that $\lambda_1 = s_1(1 - s_1)$; otherwise let $s_1 = \frac{1}{2}$.

If M is **non-compact**, then let $\eta_1, \ldots, \eta_\kappa$ be the cusps of M. Then for each $j \in \{1, \ldots, \kappa\}$, define $\frac{1}{2} \leq s_1^{(j)} \leq s_1$ from the smallest *non-cuspidal* eigenvalue, restricting to eigenfunctions which have *non-zero constant term at η_j*.

Theorem (S, 2013; for X compact: M. Burger 1990):

Fix $0 \leq \alpha < \frac{1}{2}$ and $p_0 \in M$.

Then for every $p \in X$, $T \geq 10$ and $f \in C^3(X)$ with $\|f\|_{W_3} < \infty$,

$$\frac{1}{T}\int_0^T f(p\,n(t))\,dt = \int_X f\,d\mu + O\big(\|f\|_{W_3}\big) \cdot \Big\{r^{-\frac{1}{2}}\log^3(r+2) + r^{s_1^{(j)}-1} + T^{s_1-1}\Big\} + O\big(\|f\|_{N_\alpha}\big)\,r^{-\frac{1}{2}},$$

where $r = r(p,T) := T\big/e^{\operatorname{dist}(p\,a(T))}$

with $a(T) = \begin{pmatrix} T^{1/2} & 0 \\ 0 & T^{-1/2} \end{pmatrix}$

and $\operatorname{dist}(p\,a(T))$ = distance from $\pi(p\,a(T))$ to $p_0 \in M$,

and j is the "index of the cusp which $\pi(p\,a(T))$ is near".

The implied constants depend only on Γ, α, p_0.

Note: $e^{\operatorname{dist}(p\,a(T))} \asymp \mathcal{Y}_\Gamma(p\,a(T))$, where

$$\mathcal{Y}_\Gamma(\Gamma g) = \sup\big\{\operatorname{Im} N_k W g(i) \;:\; k \in \{1,\ldots,\kappa\},\, W \in \Gamma\big\} \quad (\forall g \in G).$$

Theorem (S, 2013; for X compact: M. Burger 1990):

Fix $0 \leq \alpha < \frac{1}{2}$ and $p_0 \in M$.

Then for every $p \in X$, $T \geq 10$ and $f \in C^3(X)$ with $\|f\|_{W_3} < \infty$,

$$\frac{1}{T}\int_0^T f(p\,n(t))\,dt = \int_X f\,d\mu$$
$$+ O\big(\|f\|_{W_3}\big) \cdot \Big\{ r^{-\frac{1}{2}} \log^3(r+2) + r^{s_1^{(j)}-1} + T^{s_1-1} \Big\} + O\big(\|f\|_{N_\alpha}\big) r^{-\frac{1}{2}},$$

where $r = r(p,T) := T \big/ e^{\mathrm{dist}(p\,a(T))}$

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and $\mathrm{dist}(p\,a(T)) =$ distance from $\pi(p\,a(T))$ to $p_0 \in M$,

and j is the "index of the cusp which $\pi(p\,a(T))$ is near".

The implied constants depend only on Γ, α, p_0.

For any $p \in X$: $\lim_{T\to\infty} r(p,T) = +\infty$ iff the orbit $p\,n(\mathbb{R})$ is *not* closed.

Hence the above result is an effective version of Dani & Smillie 1984.

More technical: The function norms $\|\cdot\|_{N_\alpha}$ and $\|\cdot\|_{W_k}$

For given $0 \leq \alpha < \frac{1}{2}$: $\boxed{\|f\|_{N_\alpha} = \sup_{p\in X} |f(p)| \cdot e^{-\alpha\cdot \mathrm{dist}(p)}}$

Sobolev norm $\|\cdot\|_{W_k}$

To each $Y \in \mathfrak{g}$ corresponds a left-invariant differential operator on functions on G, and thus also on $X = \Gamma\backslash G$:

$$(Yf)(g) = \left(\frac{d}{dt} f(g \exp tY)\right)\Big|_{t=0}$$

Fix a basis Y_1, Y_2, Y_3 of $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$. For $f \in C^k(X)$, define

$$\|f\|_{W_k} := \sqrt{\sum_D \|Df\|_{L^2}^2},$$

where D runs over all monomials in Y_1, Y_2, Y_3 of degree $\leq k$.

Remark: Spectral 'pre-Theorem' in S (2013):

Proposition:

$$\frac{1}{T}\int_0^T f(p\,n(t))\,dt = \int_X f\,d\mu + O\big(\|f\|_{W_3}\big)\cdot\Big\{r^{-\frac{1}{2}}(\log T)^2 + r^{s_1^{(j)}-1} + T^{s_1-1}\Big\}.$$

This is then applied together with a more geometric argument (only needed for very special p) to get the theorem:

Theorem (restated):

$$\frac{1}{T}\int_0^T f(p\,n(t))\,dt = \int_X f\,d\mu$$
$$+\,O\big(\|f\|_{W_3}\big)\cdot\Big\{r^{-\frac{1}{2}}\log^3(r+2) + r^{s_1^{(j)}-1} + T^{s_1-1}\Big\} + O\big(\|f\|_{N_\alpha}\big)r^{-\frac{1}{2}},$$

NOTE: For X non-compact, there exist $f \in C^\infty(X)$, $f \geq 0$, with $\|f\|_{W_k} < \infty$ $(\forall k)$, and $p \in X$, such that $\limsup_{T\to\infty} \int_0^T f(p\,n(t))\,dt = \infty$.

Back to pieces of closed horocycles

Let Γ be a lattice in $G = \mathrm{PSL}(2,\mathbb{R})$ such that ∞ is a cusp of $\Gamma\backslash\mathbb{H}$ and $\Gamma_\infty = \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle$.

For $0 < y \leq 1$ and $\alpha < \beta \leq \alpha + 1$, consider

$$\frac{1}{\beta - \alpha} \int_\alpha^\beta f(x + iy, 0)\, dx = \frac{1}{T} \int_0^T f\big(p\, n(t)\big)\, dt,$$

with $p := \Gamma n(\alpha) a(y)$ and $T = \dfrac{\beta - \alpha}{y}$. Apply the previous theorem, and note that

$$r(p, T) = \frac{T}{\mathcal{Y}_\Gamma(\Gamma n(\alpha) a(y) a(T))} = \frac{T}{\mathcal{Y}_\Gamma(\Gamma n(\alpha) a(\beta - \alpha))} \geq \frac{T}{(\beta - \alpha)^{-1}} = \frac{(\beta - \alpha)^2}{y}.$$

Get:

Corollary (S, 2013):

$$\frac{1}{\beta-\alpha}\int_\alpha^\beta f(x+iy,0)\,dx = \int_X f\,d\mu + O\big(\|f\|_{N_\alpha}\big)\frac{\sqrt{y}}{\beta-\alpha}$$

$$+\,O\big(\|f\|_{W_3}\big)\left\{\frac{\sqrt{y}}{\beta-\alpha}\log\left(\frac{(\beta-\alpha)^2}{y}+2\right)+\left(\frac{\sqrt{y}}{\beta-\alpha}\right)^{2(1-s_1')}+\left(\frac{y}{\beta-\alpha}\right)^{1-s_1}\right\},$$

with $s_1' = \max\big(s_1^{(1)},\ldots,s_1^{(\kappa)}\big)$.

Here $\big[\text{error} \to 0\big] \quad\Leftrightarrow\quad \dfrac{\beta-\alpha}{\sqrt{y}} \to +\infty \quad\Leftrightarrow\quad \dfrac{T}{\sqrt{1/y}} \to +\infty$ **!**

Special case; the *full* closed horocycle; $\beta = \alpha + 1$:

$$\int_0^1 f(x+iy,0)\,dx = \int_X f\,d\mu + O_f\Big(y^{\frac{1}{2}}\log(y^{-1}) + y^{1-s_1}\Big).$$

Trivially: Can replace y^{1-s_1} by $y^{1-s_1'}$.

– Sarnak 1981: Discussed *precise correction terms*, and got error $o(y^{1/2})$.

– Zagier 1979: For $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$; get $O_f(y^{-\frac{3}{4}+\varepsilon})$ for all $f \in C_c^\infty(M)$ iff the Riemann Hypothesis holds!

Spectral theory of the Laplace operator on $M = \Gamma\backslash\mathbb{H}$

Let $\Delta = -y^2\big(\partial_x^2 + \partial_y^2\big)$, the Laplace-Beltrami operator on $\mathbb{H}$ and on $M = \Gamma\backslash\mathbb{H}$.

Let

$$\phi_0, \phi_1, \phi_2, \ldots \in L^2(M)$$

be the discrete eigenfunctions of Δ on M, with

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$$

the corresponding eigenvalues.

We take $\phi_0, \phi_1, \ldots$ to be ON, i.e.

$$\langle \phi_j, \phi_k \rangle = \int_{\Gamma\backslash\mathbb{H}} \phi_j(z)\overline{\phi_k(z)}\, dA(z) = \delta_{j-k}.$$

(Here $dA(z) = \dfrac{dx\, dy}{y^2}$, the hyperbolic area measure.)

If M is compact then $\phi_0, \phi_1, \phi_2, \ldots$ form a Hilbert basis of $L^2(M)$.

Spectral theory of Δ on $M = \Gamma\backslash\mathbb{H}$ – for M non-compact

Let $\eta_1 = \infty, \eta_2, \ldots, \eta_\kappa \in \partial\mathbb{H}$ be representatives of the cusps of M. Choose $N_1, \ldots, N_\kappa \in G$ so that $N_k(\eta_k) = \infty$ and $\Gamma_{\eta_k} = N_k^{-1} \left\langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right) \right\rangle N_k$. (Take $N_1 = I_2$.)

For each $k \in \{1, \ldots, \kappa\}$, let $E_k(z, s)$ be the Eisenstein series associated to the cusp η_k. Thus:

$$E_k(z,s) = \sum_{\gamma \in \Gamma_{\eta_k}\backslash\Gamma} \left(\operatorname{Im} N_k \gamma z\right)^s \qquad (\operatorname{Re} s > 1)$$

$E_k(z, s)$ has a meromorphic continuation to s in all $\mathbb{C}$, and

$$E_k(\gamma z, s) = E_k(z, s), \qquad \forall \gamma \in \Gamma,\ z \in \mathbb{H};$$

$$E_k(z, s) \text{ is } C^\infty \qquad \text{on } \mathbb{H} \times (\mathbb{C} \setminus \{\text{poles}\});$$

$$\Delta_z E_k(z, s) = s(1-s) E_k(z, s) \qquad \text{on } \mathbb{H} \times (\mathbb{C} \setminus \{\text{poles}\});$$

Also $E_k(z, s)$ is holomorphic on the line $\operatorname{Re} s = \frac{1}{2}$.

Spectral theory of Δ on $M = \Gamma\backslash\mathbb{H}$ – for M non-compact

Now any $f \in L^2(M)$ has the spectral expansion

$$\boxed{f = \sum_{m\geq 0} d_m \phi_m + \sum_{k=1}^{\kappa} \int_0^\infty g_k(r) E_k(\cdot, \tfrac{1}{2} + ir)\, dr} \qquad (*)$$

where

$$d_m = \langle f, \phi_m \rangle; \qquad g_k(r) = \frac{1}{2\pi} \int_M f(z) \overline{E_k(z, \tfrac{1}{2} + ir)}\, d\mu(z).$$

(“$\int_0^\infty \cdots$” stands for a limit in $L^2(M)$, and “$\int_M \cdots$” for a limit in $L^2(\mathbb{R}_{>0})$.)

Also:

$$\int_M |f(z)|^2\, d\mu(z) = \sum_{m\geq 0} |d_m|^2 + 2\pi \sum_{k=1}^{\kappa} \int_0^\infty |g_k(r)|^2\, dr.$$

For any $f \in \mathrm{C}^2(M)$ such that $f \in \mathrm{L}^2(M)$ and $\Delta f \in \mathrm{L}^2(M)$: $(*)$ holds pointwise, with uniform absolute convergence over z in compact subsets of M.

Ergodic average along a piece of a closed horocycle

Using the spectral expansion (for $f \in C^2(M)$ with $f, Df \in L^2(M)$):

$$f(z) = \sum_{m \geq 0} d_m \phi_m(z) + \sum_{k=1}^{\kappa} \int_0^\infty g_k(r) E_k(z, \tfrac{1}{2} + ir)\, dr,$$

we now wish to study the ergodic average

$$\frac{1}{\beta - \alpha} \int_\alpha^\beta f(x + iy)\, dx \qquad \text{as } y \to 0.$$

It is

$$= \sum_{m \geq 0} d_m \left(\frac{1}{\beta - \alpha} \int_\alpha^\beta \phi_m(x + iy)\, dx \right)$$
$$+ \sum_{k=1}^{\kappa} \int_0^\infty g_k(r) \left(\frac{1}{\beta - \alpha} \int_\alpha^\beta E_k(x + iy, \tfrac{1}{2} + ir)\, dx \right) dr,$$

Here

$$\frac{d_0}{\beta - \alpha} \int_\alpha^\beta \phi_0(x + iy)\, dx \boxed{= \frac{1}{A(M)} \int_M f\, dA} \boxed{= \int_X f\, d\mu}$$

(since $\phi_0(z) \equiv A(M)^{-1/2}$ and $d_0 = \langle f, \phi_0 \rangle = A(M)^{-1/2} \int_M f\, dA$).

“Morally” sufficient:

For $\boxed{\phi = \phi_m}$ (some m) *or* $\boxed{\phi = E_k(\cdot, \frac{1}{2} + ir)}$ (some k and some $r \in \mathbb{R}_{>0}$), prove

$$\frac{1}{\beta - \alpha} \int_\alpha^\beta \phi(x + iy)\, dx \to 0 \qquad \text{as } \; y \to 0.$$

Fourier expansion of $\phi(z)$:

$$\boxed{\phi(x+iy) = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} y^s + c_0 y^{1-s} + \sum_{n \in \mathbb{Z} \setminus \{0\}} c_n \sqrt{y}\, K_{ir}\big(2\pi |n| y\big)\, e(nx)}$$

Here:

– $r = r_m \in \mathbb{R}_{\geq 0} \cup i(-\frac{1}{2}, 0)$ in the discrete case; also $s = \frac{1}{2} + ir$.
Thus $\Delta \phi \equiv (\frac{1}{4} + r^2)\phi = s(1-s)\phi$.

– $e(nx) = e^{2\pi i n x}$

– $c_n = c_n^{(k,r)}$ resp $c_n = c_n^{(m)}$.

– $K_{ir}(u) = \int_0^\infty e^{-u \cosh(t)} \cos(rt)\, dt$, the K-Bessel function.
It satisfies $\big(u^2 \partial_u^2 + u \partial_u - u^2 + r^2\big) K_{ir}(u) = 0$.

Using

$$\boxed{\phi(x+iy) = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} y^s + c_0 y^{1-s} + \sum_{n\in\mathbb{Z}\setminus\{0\}} c_n \sqrt{y}\, K_{ir}\big(2\pi|n|y\big)\, e(nx)}$$

get:

$$\frac{1}{\beta-\alpha}\int_\alpha^\beta \phi(x+iy)\,dx = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} y^s + c_0 y^{1-s} + \frac{1}{\beta-\alpha}\sum_{n\neq 0} c_n \sqrt{y}\, K_{ir}\big(2\pi|n|y\big)\, \frac{e(n\beta)-e(n\alpha)}{2\pi i n}$$

Here use

$$\sum_{1\leq|n|\leq N} |c_n|^2 \ll_r N\log N \qquad \text{as } N\to\infty$$

("Rankin-Selberg type bound"), and IF $r \in \mathbb{R}_{\geq 0}$:

$$|K_{ir}(u)| \ll_r e^{-u}\log(2+u^{-1}) \qquad \forall u>0,$$

and

$$\left|\frac{e(n\beta)-e(n\alpha)}{2\pi i n}\right| \ll \min\left(|\beta-\alpha|, \frac{1}{|n|}\right).$$

Get, IF $r \in \mathbb{R}_{\geq 0}$:

$$\frac{1}{\beta-\alpha}\int_\alpha^\beta \phi(x+iy)\,dx \ll_{r,\varepsilon} \sqrt{y} + \frac{\sqrt{y}}{\beta-\alpha}\sum_{n\neq 0}|c_n|e^{-2\pi|n|y}\left(|n|y\right)^{-\varepsilon}\cdot|n|^{-1}$$

$$= \sqrt{y} + \frac{y^{\frac{1}{2}-\varepsilon}}{\beta-\alpha}\int_{1-}^{\infty} e^{-2\pi yx}x^{-1-\varepsilon}\,dS(x),$$

where

$$S(x) := \sum_{0<|n|\leq x}|c_n|.$$

Ranking-Selberg bound & Cauchy–Schwarz

$$\Rightarrow \quad S(x) \ll x\sqrt{\log x} \ll_\varepsilon x^{1+\frac{\varepsilon}{2}} \qquad \text{as } x\to\infty.$$

Hence get:

$$\frac{1}{\beta-\alpha}\int_\alpha^\beta \phi(x+iy)\,dx \ll_{r,\varepsilon} \sqrt{y} + \frac{y^{\frac{1}{2}-\varepsilon}}{\beta-\alpha}\int_1^\infty (y+x^{-1})e^{-2\pi yx}x^{-1-\varepsilon}S(x)\,dx$$

$$\ll_\varepsilon \sqrt{y} + \frac{y^{\frac{1}{2}-\varepsilon}}{\beta-\alpha}\left(\int_1^{y^{-1}} x^{-1-\frac{\varepsilon}{2}}\,dx + \int_{y^{-1}}^\infty ye^{-2\pi yx}\,dx\right)$$

$$\ll_\varepsilon \frac{y^{\frac{1}{2}-\varepsilon}}{\beta-\alpha}.$$

(Working more carefully with $S(x) \ll x\sqrt{\log x}$, get $\cdots \ll_r \frac{\sqrt{y}}{\beta-\alpha}(\log(1+y^{-1}))^{5/2}$.)

Uniformity wrt. the eigenvalue – key ingredients for $\phi = E_k(\cdot, \frac{1}{2} + ir)$

Uniform version of the Rankin-Selberg bound:

$$\sum_{1 \le |n| \le N} |c_n|^2 \ll e^{\pi r}(N + r)\Big(\omega(r) + \log\Big(\frac{2N}{r+1} + r\Big)\Big).$$

Here $\omega(r)$ is a "spectral majorant", which satisfies $\omega(r) \ge 1$ and $\int_0^T \omega(r)\,dr \ll T^2$ as $T \to \infty$. (Also $\operatorname{Tr}\Big(\Phi'(\frac{1}{2} + ir)\Phi(\frac{1}{2} + ir)^{-1}\Big) \ll \omega(r)$.)

Uniform bound on $K_{ir}(u)$ for $r \ge 1$, $u > 0$:

$$|K_{ir}(u)| \ll e^{-\frac{\pi}{2} r} r^{-\frac{1}{3}} \min\Big(1, e^{\frac{\pi}{2} r - u}\Big).$$

These two together lead to (for $r \ge 1$):

$$\frac{1}{\beta - \alpha} \int_\alpha^\beta E_k\big(x + iy, \tfrac{1}{2} + ir\big)\,dx \ll_\varepsilon r^{\frac{1}{6} + \varepsilon} \sqrt{\omega(r)} \cdot \frac{y^{\frac{1}{2} - \varepsilon}}{\beta - \alpha}$$

Hence for the *total* contr. from Eisenstein series to $\frac{1}{\beta-\alpha}\int_\alpha^\beta f(x+iy)\,dx$:

$$\sum_{k=1}^{\kappa}\int_0^\infty g_k(r)\frac{1}{\beta-\alpha}\int_\alpha^\beta E_k\Big(x+iy,\tfrac{1}{2}+ir\Big)\,dx\,dr$$

$$\ll \sum_{k=1}^{\kappa}\int_0^\infty \big|g_k(r)\big|\cdot(r+1)^{\frac{1}{6}+\varepsilon}\sqrt{\omega(r)}\,dr\cdot\frac{y^{\frac{1}{2}-\varepsilon}}{\beta-\alpha}$$

$$\ll \sum_{k=1}^{\kappa}\sqrt{\int_0^\infty |g_k(r)|^2(r+1)^4\,dr}\sqrt{\int_0^\infty (r+1)^{\frac{1}{3}+2\varepsilon-4}\omega(r)\,dr}\cdot\frac{y^{\frac{1}{2}-\varepsilon}}{\beta-\alpha}$$

$$\ll \Big(\|f\|_{\mathrm{L}^2}+\|\Delta f\|_{\mathrm{L}^2}\Big)\cdot\frac{y^{\frac{1}{2}-\varepsilon}}{\beta-\alpha}.$$

Contributions from small eigenvalues

Fix $\boxed{\phi = \phi_m}$ (some m); and assume $0 < \lambda_m < \frac{1}{2}$. Write $\lambda_m = s(1-s)$ with $\frac{1}{2} < s < 1$.

$$\boxed{\phi(x+iy) = c_0 y^{1-s} + \sum_{n \in \mathbb{Z} \setminus \{0\}} c_n \sqrt{y}\, K_{s-\frac{1}{2}}(2\pi|n|y)\, e(nx)}$$

Using $\sum_{1 \le |n| \le N} |c_n|^2 \ll_r N \log N$ and $|K_{s-\frac{1}{2}}(u)| \ll u^{\frac{1}{2}-s} e^{-u}$, get only

$$\frac{1}{\beta - \alpha} \int_\alpha^\beta \phi(x+iy)\, dx \ll y^{1-s} (\beta - \alpha)^{s-\frac{3}{2}},$$

which is not good enough!

USE INSTEAD: Bound on linear forms (S, '04);

$$\boxed{\sum_{n=1}^{N} c_n e(n\nu) = O\big(N^{\frac{3}{2}-s}\big), \qquad \forall N \ge 1,\ \nu \in \mathbb{R}}$$

If ϕ is a *cusp form* then $\cdots = O_\varepsilon\big(N^{\frac{1}{2}+\varepsilon}\big)$ (Hafner, '85).

As before,

$$\frac{1}{\beta-\alpha}\int_\alpha^\beta \phi(x+iy)\,dx = c_0 y^{1-s} + \frac{1}{\beta-\alpha}\sum_{n\neq 0} c_n\sqrt{y}\,K_{s-\frac12}(2\pi|n|y)\,\frac{e(n\beta)-e(n\alpha)}{2\pi i n}.$$

Writing $\boxed{\delta := \beta-\alpha}$ and $\boxed{S_\nu(Y) := \sum_{1\le n\le Y} c_n e(n\nu)}$, we have

$$\frac{1}{\beta-\alpha}\sum_{n=1}^{\infty} c_n\sqrt{y}\,K_{s-\frac12}(2\pi n y)\,\frac{e(n\beta)-e(n\alpha)}{n}$$

$$= \frac{\sqrt{y}}{\delta}\sum_{1\le n\le \delta^{-1}} K_{s-\frac12}(2\pi n y)\frac{e(n\delta)-1}{n}\cdot c_n e(n\alpha)$$

$$+ \frac{\sqrt{y}}{\delta}\sum_{n>\delta^{-1}} K_{s-\frac12}(2\pi n y)\frac{1}{n}\cdot c_n e(n\beta) - \Big[\text{same with } c_n e(n\alpha)\Big]$$

$$= \frac{\sqrt{y}}{\delta}\int_{1-}^{\delta^{-1}} K_{s-\frac12}(2\pi x y)\frac{e(x\delta)-1}{x}\cdot dS_\alpha(x)$$

$$+ \frac{\sqrt{y}}{\delta}\int_{\delta^{-1}}^{\infty} K_{s-\frac12}(2\pi x y)\frac{1}{x}\cdot dS_\beta(x) - \Big[\text{same with } dS_\alpha(x)\Big]$$

Set

$$f(x) = K_{s-\frac{1}{2}}(2\pi xy)\frac{e(x\delta)-1}{x}; \qquad g(x) = K_{s-\frac{1}{2}}(2\pi xy)\frac{1}{x},$$

so that the above is

$$\frac{\sqrt{y}}{\delta}\left(\int_{1-}^{\delta^{-1}} f(x)\,dS_\alpha(x) + \int_{\delta^{-1}}^{\infty} g(x)\,dS_\beta(x) - \int_{\delta^{-1}}^{\infty} g(x)\,dS_\alpha(x)\right)$$

$$= \frac{\sqrt{y}}{\delta}\Bigg(f(\delta^{-1})S_\alpha(\delta^{-1}) - g(\delta^{-1})S_\beta(\delta^{-1}) + g(\delta^{-1})S_\alpha(\delta^{-1})$$

$$- \int_{1}^{\delta^{-1}} f'(x)\,S_\alpha(x)\,dx - \int_{\delta^{-1}}^{\infty} g'(x)\,S_\beta(x)\,dx + \int_{\delta^{-1}}^{\infty} g'(x)\,S_\beta(x)\,dx\Bigg)$$

Using now

$$|K_{s-\frac{1}{2}}(u)| \ll \left\{\begin{matrix} u^{\frac{1}{2}-s} & (u \le 1) \\ u^{-\frac{1}{2}}e^{-u} & (u > 1)\end{matrix}\right\} \ll u^{\frac{1}{2}-s}e^{-\frac{1}{2}u}$$

and

$$|K'_{s-\frac{1}{2}}(u)| \ll \left\{\begin{matrix} u^{-\frac{1}{2}-s} & (u \le 1) \\ u^{-\frac{1}{2}}e^{-u} & (u > 1)\end{matrix}\right\} \ll u^{-\frac{1}{2}-s}e^{-\frac{1}{2}u},$$

and $y \le \delta \le 1$, we have

$$|f(\delta^{-1})|,\ |g(\delta^{-1})| \ll \delta^{\frac{1}{2}+s} y^{\frac{1}{2}-s};$$

$$|f'(x)| \ll \delta y^{\frac{1}{2}-s} x^{-\frac{1}{2}-s} \qquad \text{for } 0 < x \le \delta^{-1};$$

$$|g'(x)| \ll y^{\frac{1}{2}-s} x^{-\frac{3}{2}-s} e^{-\pi y x} \qquad \text{for } x \ge \delta^{-1};$$

Using these and $S_\nu(x) \ll x^{\frac{3}{2}-s}$ $(\forall x \ge 1)$, we finally get:

$$\boxed{\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta \phi(x+iy)\,dx\right| \ll y^{1-s}\delta^{2(s-1)} = \left(\frac{\sqrt{y}}{\beta-\alpha}\right)^{2(1-s)}}$$

If ϕ is a *cusp form*, then using Hafner's bound, $S_\nu(x) \ll x^{\frac{1}{2}+\varepsilon}$, we get the stronger bound:

$$\boxed{\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta \phi(x+iy)\,dx\right| \ll y^{1-s-\varepsilon}\delta^{s-1} = \left(\frac{y}{\beta-\alpha}\right)^{1-s} y^{-\varepsilon}}$$

The above analysis leads to the following (mainly weaker!) variant of the Theorem on p. 15:

Theorem (S, '04): Let Γ be a lattice in $G = \mathrm{PSL}(2,\mathbb{R})$ such that ∞ is a cusp of $\Gamma\backslash\mathbb{H}$ and $\Gamma_\infty = \left\langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \right\rangle$.

If there exist small eigenvalues $0 < \lambda < \frac{1}{4}$ of the Laplace operator on $\Gamma\backslash\mathbb{H}$, let λ_1 be the smallest of these and define $\frac{1}{2} < s_1 < 1$ so that $\lambda_1 = s_1(1-s_1)$; otherwise let $s_1 = \frac{1}{2}$.

Similarly define $\frac{1}{2} \leq s_1' \leq s_1$ from the smallest *non-cuspidal* eigenvalue.

Let $f \in \mathrm{C}^2(M)$ with $f, \Delta f \in \mathrm{L}^2(M)$, and let $0 < y \leq 1$ and $\alpha < \beta \leq \alpha + 1$. Then:

$$\frac{1}{\beta-\alpha}\int_\alpha^\beta f(x+iy)\,dx = \frac{1}{A(M)}\int_M f\,dA + O\Big(\|f\|_{\mathrm{L}^2} + \|\Delta f\|_{\mathrm{L}^2}\Big)\cdot\frac{y^{\frac{1}{2}-\varepsilon}}{\beta-\alpha}$$
$$+O\Big(\|f\|_{\mathrm{L}^2}\Big)\left\{\Big(\frac{\sqrt{y}}{\beta-\alpha}\Big)^{2(1-s_1')} + \Big(\frac{y}{\beta-\alpha}\Big)^{1-s_1} y^{-\varepsilon}\right\}$$

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