#3. Equidistribution in the space of 2-dimensional affine lattices

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Set $\quad G = \mathrm{ASL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2,$

$$\text{with } (M_1,\vec{v}_1)(M_2,\vec{v}_2) = (M_1M_2, \vec{v}_1M_2 + \vec{v}_2).$$

G acts from the right on $\mathbb{R}^2$ by affine linear maps:

$$\vec{y}(M,\vec{v}) = \vec{y}M + \vec{v}, \qquad \text{for } (M,\vec{v}) \in G,\ \vec{y} \in \mathbb{R}^2.$$

Set $\quad \Gamma = \mathrm{ASL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2;$

$X = \Gamma\backslash G;$

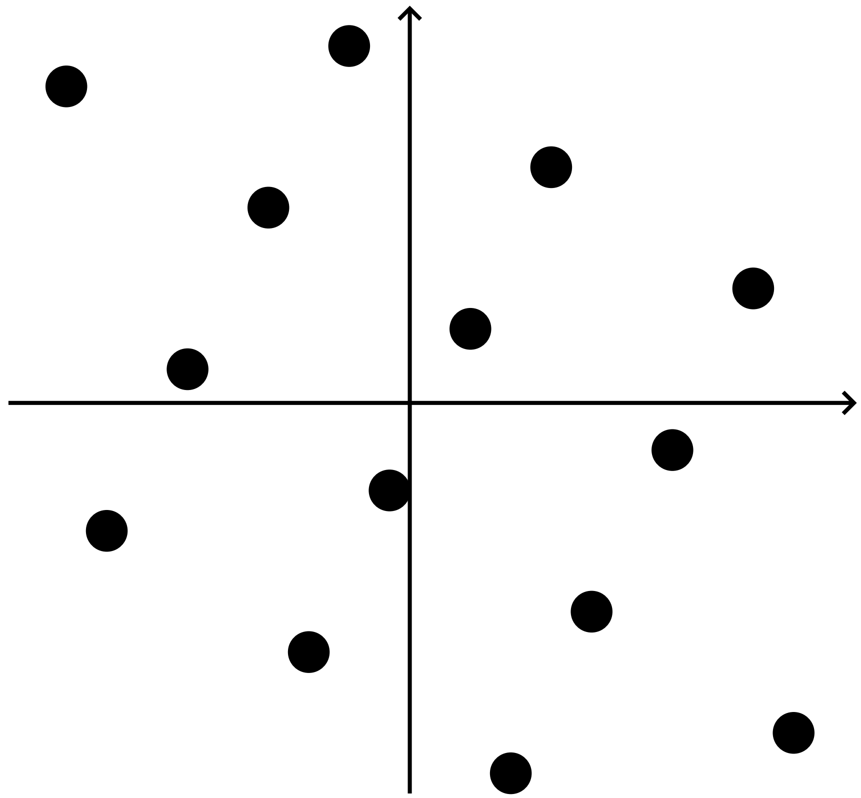
$\mu =$ Haar measure on G; $\mu(X) = 1$.

X is a torus bundle over $Y = \Gamma_0\backslash G_0 = \mathrm{SL}(2,\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{R})$.

Projection map $X \to Y$, $\quad \Gamma(M,\vec{v}) \mapsto \Gamma_0 M$.

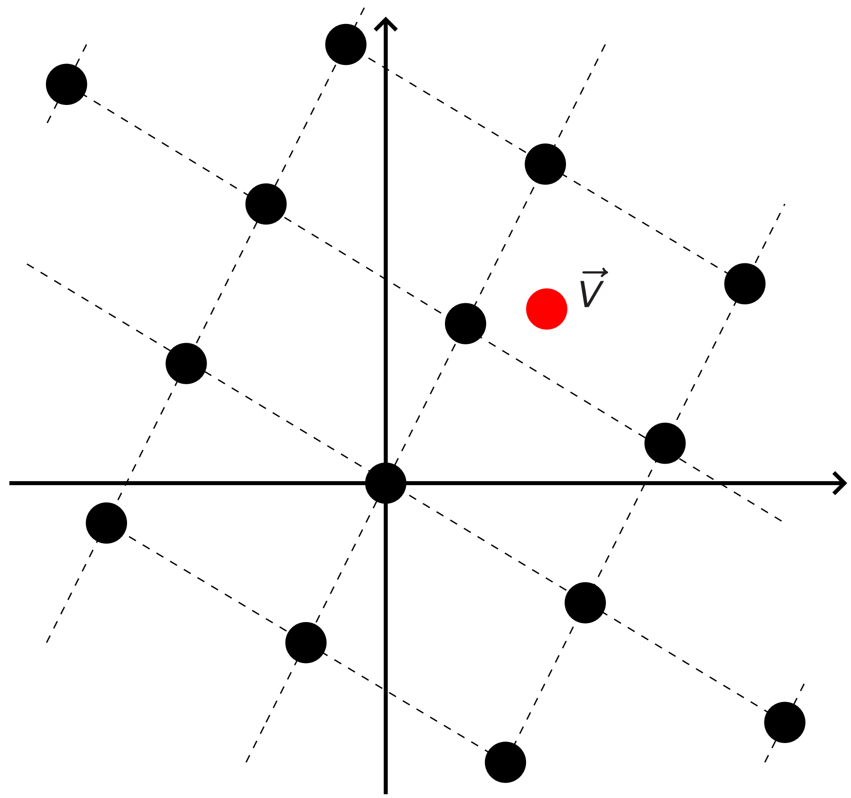
Fiber over $\Gamma_0 M$: $\quad \{\Gamma(1_2,\vec{v})M \ :\ \vec{v} \in \mathbb{Z}^2\backslash\mathbb{R}^2\}$.

X can be identified with the space of *affine unimodular lattices in* $\mathbb{R}^2$, through $\Gamma g \mapsto \mathbb{Z}^2 g$, i.e. $\Gamma(M, \vec{v}) \mapsto \mathbb{Z}^2 M + \vec{v}$



OR: X may be identified with the space of pairs

$$\left\{ (L, \vec{v}) \ : \ L \textit{ unimodular lattice in } \mathbb{R}^2\textit{, and } \ \vec{v} \in \mathbb{R}^2 \bmod L \right\}.$$



Unipotent orbits in X

Set

$$u(x) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, (0,0) \right); \qquad \text{an Ad-unipotent subgroup of } G.$$

But there exist other Ad-unipotent subgroups not conjugate to $u(\mathbb{R})$:

$$x \mapsto \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, (2\alpha x, \alpha x^2) \right) \qquad \text{any fixed } \alpha \neq 0.$$

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$$a(y) = \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix}.$$

For any $\vec{\xi} \in \mathbb{R}^2$, $y \in \mathbb{R}_{>0}$, consider the $u(\mathbb{R})$-orbit $\boxed{\Gamma\big(1_2, \vec{\xi}\big) a(y) u(\mathbb{R})}$ in X.

Note $u(x)a(y) = a(y)u(x/y)$; hence $\boxed{\Gamma\big(1_2, \vec{\xi}\big) a(y) u(\mathbb{R}) = \Gamma\big(1_2, \vec{\xi}\big) U(\mathbb{R}) a(y)}$,

and: projects to a *closed* horocycle in $Y = \mathrm{SL}(2,\mathbb{Z}) \backslash \mathrm{SL}(2,\mathbb{R})$, length $1/y$.

Do the $a(y)$-push-forwards of any fixed piece $\big\{\Gamma(1_2, \vec{\xi}) u(x) \ : \ \alpha < x < \beta\big\}$ become asymptotically equidistributed in (X, μ) as $y \to 0$?

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Do the $a(y)$-push-forwards of any fixed piece $\{\Gamma(1_2, \vec{\xi})u(x) \,:\, \alpha < x < \beta\}$ become asymptotically equidistributed in (X, μ) as $y \to 0$?

Answer: Yes if.f. $\vec{\xi} \notin \mathbb{Q}^2$.

So, we have, for any $\vec{\xi} \in \mathbb{R}^2 \setminus \mathbb{Q}^2$, $\alpha < \beta$, $f \in C_b(X)$:

$$\lim_{y \to 0} \frac{1}{\beta - \alpha} \int_\alpha^\beta f\Big(\Gamma(1_2, \vec{\xi})\, u(x)\, a(y)\Big)\, dx = \int_X f\, d\mu.$$

(Consequence of Ratner's measure classification; special case of Shah 1996; cf. also Elkies & McMullen 2004.)

On the other hand, if $\vec{\xi} \in \mathbb{Q}^2$, let $q = d(\vec{\xi})$, the "denominator" of $\vec{\xi}$.
Then $\Big\{\Gamma(1_2, \vec{\xi})\, u(x)\, a(y) \;:\; \alpha < x < \beta\Big\}$ is a piece of a closed horocycle in the 3-dimensional homogeneous subspace

$$X_q := \Big\{\Gamma(1_2, \vec{v})M \;:\; M \in \mathrm{SL}(2, \mathbb{R}),\ \vec{v} \in \mathbb{Q}^2,\ d(\vec{v}) = q\Big\}$$

$$\cong \Gamma_1(q) \backslash \mathrm{SL}(2, \mathbb{R}),$$

and we have

$$\lim_{y \to 0} \frac{1}{\beta - \alpha} \int_\alpha^\beta f\Big(\Gamma(1_2, \vec{\xi})\, u(x)\, a(y)\Big)\, dx = \int_{X_q} f\, d\mu_q.$$

Effective result – using Margulis' thickening technique?

We wish to prove that, if $\vec{\xi} \notin \mathbb{Q}^2$, then

$$\frac{1}{\beta-\alpha}\int_\alpha^\beta f\Big(\Gamma(1_2,\vec{\xi})\,u(x)\,a(y)\Big)\,dx \to \int_X f\,d\mu, \qquad \text{as } \ y\to 0,$$

with an explicit rate!

Note that the flow $a(\mathbb{R}_+^\times)$ on $X=\Gamma\backslash G$ is HYPERBOLIC, with

unstable manifolds (for $y\to 0$) generated by "$\left(\begin{pmatrix}1 & \mathbb{R}\\ 0 & 1\end{pmatrix},(0,\mathbb{R})\right)$"

(*different* rates of expansion!)

stable manifolds generated by "$\left(\begin{pmatrix}1 & 0\\ \mathbb{R} & 1\end{pmatrix},(\mathbb{R},0)\right)$"

(*different* rates of contraction!).

The thickening technique could certainly be applied to study a *2-dimensional* average,

$$\frac{1}{|I|\,|J|}\int_I\int_J f\Big(\Gamma(1_2,\vec{\xi})\,u(x)\,a(y)\Big)\,dx\,d\xi_2 \qquad \text{as } \ y\to 0;$$

however for our *1-dimensional* average it does not work.

Connecting with Dolgopyat's (more standard) notation

Let $\widetilde{G} = \Big\{(M, \vec{v}) \;:\; M \in \mathrm{SL}(2,\mathbb{R}),\; \vec{v} \text{ column vector } \in \mathbb{R}^2\Big\}$, with multiplication $(M_1, \vec{v}_1)(M_2, \vec{v}_2) = (M_1 M_2, M_1 \vec{v}_2 + \vec{v}_1)$.

Action $\widetilde{G} \times \mathbb{R}^2 \to \mathbb{R}^2$: $(M, \vec{v})\,\boldsymbol{u} := M\boldsymbol{u} + \vec{v}$.

Let $\widetilde{\Gamma} = \widetilde{G}(\mathbb{Z})$.

Key claim: For (x, α) uniformly random in $(\mathbb{R}/\mathbb{Z})^2$, the distribution of the random affine lattice

$$L_{N,\alpha}^{(x)} = \begin{pmatrix} N & \\ & 1/N \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \left(\mathbb{Z}^2 + \begin{pmatrix} x \\ 0 \end{pmatrix} \right) = \begin{pmatrix} N & \\ & 1/N \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \left(1_2, \begin{pmatrix} x \\ 0 \end{pmatrix} \right) \mathbb{Z}^2$$

tends to "Haar" on $\widetilde{G}/\widetilde{\Gamma}$, as $N \to \infty$.

Anti-isomorphism:

$$J : \widetilde{G} \xrightarrow{\sim} G; \qquad J\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = \left(\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, (v_2, -v_1)\right)$$

This comes from embedding $\widetilde{G}$ and G in $\mathrm{SL}(3,\mathbb{R})$ through $(M,\vec{v}) \mapsto \begin{pmatrix} M & \vec{v} \\ 0 & 1 \end{pmatrix}$, resp., $(M,\vec{v}) \mapsto \begin{pmatrix} M & 0 \\ \vec{v} & 1 \end{pmatrix}$, and then consider the anti-automorphism $g \mapsto \begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix} g^T \begin{pmatrix} \omega^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ of $\mathrm{SL}(3,\mathbb{R})$.

Using J, we get an identification map $\widetilde{G}/\widetilde{\Gamma} \xrightarrow{\sim} X = G/\Gamma, \quad \widetilde{\Gamma} g \mapsto J(g)/\Gamma$.

This maps $L_{N,\alpha}^{(x)}$ in $\widetilde{G}/\widetilde{\Gamma}$ to

$$\Gamma\big(1_2,(0,-x)\big)\begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/N & 0 \\ 0 & N \end{pmatrix} = \Gamma\big(1_2,(0,-x)\big)\,u(-\alpha)a(y) \quad \text{in } X = \Gamma\backslash G,$$

with $y = N^{-2}$. Hence we get back the equidistribution statement for (x,α) uniformly random in $(\mathbb{R}/\mathbb{Z})^2$.

But *also*, the more difficult (Shah 1996) equidistribution statement gives:

Theorem: Let $V_N(x, \alpha, c) = \#\Big\{1 \leq n \leq N \ : \ x + n\alpha \in [-c/N, c/N]\Big\}$.

Then for any fixed $c > 0$, fixed irrational x, and α uniformly random in any fixed interval in $\mathbb{R}/\mathbb{Z}$, $\quad V_N(x, \alpha, c) \overset{N\to\infty}{\Longrightarrow} \mathcal{Y}_{(c)}$.

Effective equidistribution, for 1-dim unipotent in $X = \Gamma\backslash G$

Theorem 1 (S, '15) For any $0 < y < 1$, $\vec{\xi} \in \mathbb{R}^2$, $\alpha < \beta$, $f \in C_b^8(X)$:

$$\left| \frac{1}{\beta - \alpha} \int_\alpha^\beta f\Big(\Gamma(1_2, \vec{\xi}\,)\, u(x)\, a(y)\Big)\, dx - \int_X f\, d\mu \right|$$

$$\ll_\varepsilon \|f\|_{C_b^8} \frac{L}{\beta - \alpha} \Big(b_{\vec{\xi}, L}(y) + y^{\frac{1}{4}} \Big)^{1-\varepsilon},$$

where $L = \max(1, |\alpha|, |\beta|)$ and

$$b_{\vec{\xi}, L}(y) = \max_{q \in \mathbb{Z}^+} \min\Big(\frac{1}{q^2}, \frac{\sqrt{y}}{L q \langle q\xi_1 \rangle}, \frac{\sqrt{y}}{q \langle q\xi_2 \rangle} \Big).$$

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(Here $\langle \cdot \rangle$ =distance to nearest integer.)

$\Big[\, b_{\vec{\xi},L}(y) \xrightarrow{y \to 0} 0 \,\Big] \iff \vec{\xi} \notin \mathbb{Q}^2.$

$\Big[\, b_{\vec{\xi},L}(y) \ll y^\delta$ as $y \to 0 \,\Big]$, if $\vec{\xi}$ has Dioph. type K and $\delta = \min(\frac{1}{2}, K^{-1})$.

(Def: $\vec{\xi}$ is of Diophantine type K iff $\exists c > 0$: $\forall q \in \mathbb{Z}^+$: $\forall \vec{m} \in \mathbb{Z}^2$:

$$\|\vec{\xi} - q^{-1}\vec{m}\| > cq^{-K}.)$$

Consequence for orbits of a FIXED point

Theorem 2 (S, '15) For any $g \in G$, $T \geq 2$, $f \in C_b^8(X)$:

$$\left| \frac{1}{T} \int_0^T f(\Gamma g u(t))\, dt - \int_X f\, d\mu \right| \ll_\varepsilon \|f\|_{C_b^8} \Big(y_g(T)^{\frac{1}{4}} + b_g(T) \Big)^{\frac{1}{2}-\varepsilon},$$

where

$$y_g(T) = T^{-1} \cdot \mathrm{Height}_Y(\Gamma g a(T))$$

and

$$b_g(T) = \inf \Big\{ \delta > 0 \ : \ \Big(\forall q \in \mathbb{Z}^+_{\leq \delta^{-1/2}} \ : \ (q^{-1}\mathbb{Z}^2) g \cap \frac{1}{\delta q^2} \mathfrak{R}_T = \emptyset \Big) \Big\}$$

where $\mathfrak{R}_T = [-T^{-1}, T^{-1}] \times [-1, 1]$.

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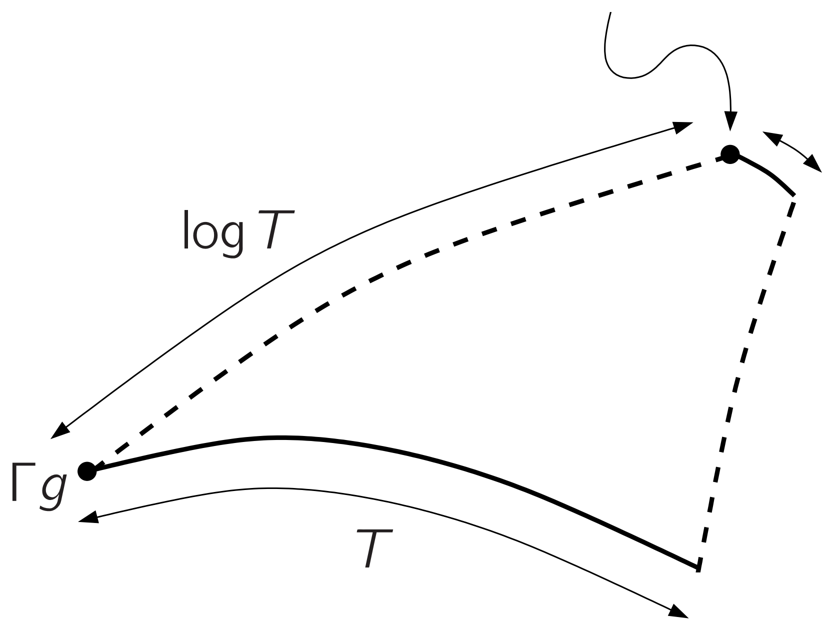
$y_g(T) = T^{-1} \cdot \mathrm{Height}_Y(\Gamma g a(T))$ (cuspidal height in Y).



$\left[\; y_g(T) \xrightarrow{T \to \infty} 0 \;\right]$

$\Longleftrightarrow$

$\pi(\Gamma g u(\mathbb{R}))$ is not closed in Y.

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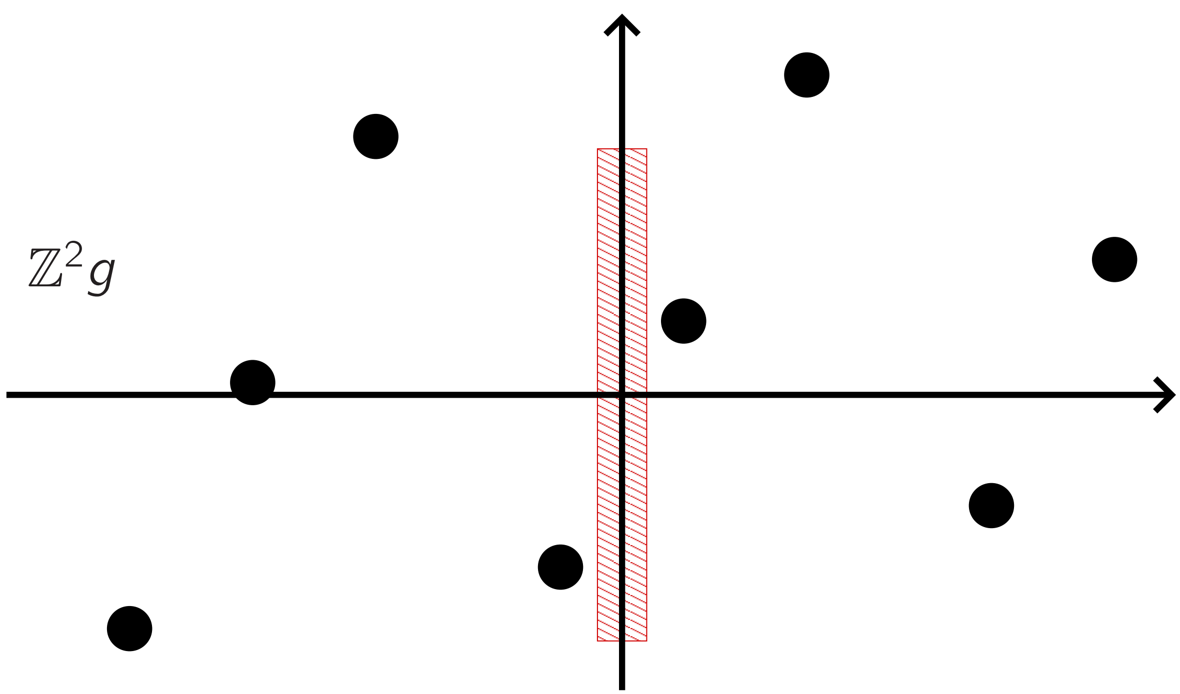

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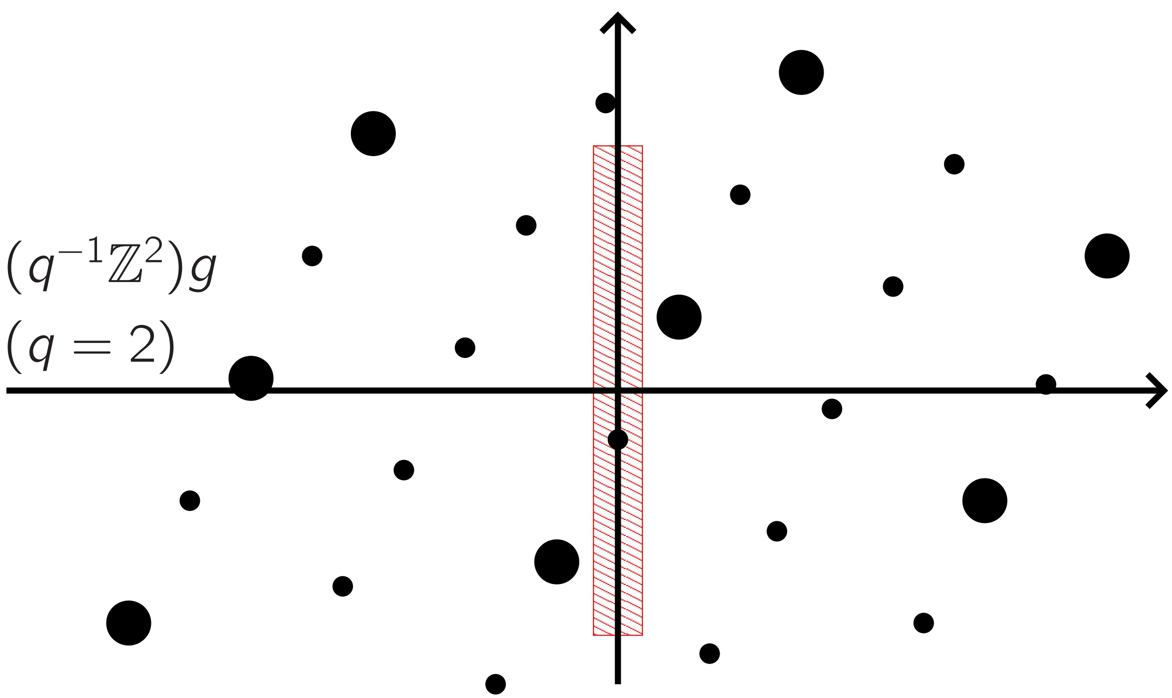

$$\Big[\ b_g(T) \xrightarrow{T \to \infty} 0 \ \Big]$$

$$\Longleftrightarrow$$

$$\mathbb{Q}^2 g \cap \big(\{0\} \times \mathbb{R} \big) = \emptyset.$$

Consequence for orbits of a FIXED point

Theorem 2 (S, '15) For any $g \in G$, $T \geq 2$, $f \in C_b^8(X)$:

$$\left| \frac{1}{T} \int_0^T f(\Gamma g u(t))\, dt - \int_X f\, d\mu \right| \ll_\varepsilon \|f\|_{C_b^8} \Big(y_g(T)^{\frac{1}{4}} + b_g(T) \Big)^{\frac{1}{2} - \varepsilon}.$$

If $y_g(T) \not\to 0$, $b_g(T) \not\to 0$: $\Gamma g u(\mathbb{R})$ closed.

If $y_g(T) \to 0$, $b_g(T) \not\to 0$: $\overline{\Gamma g u(\mathbb{R})}$ "=" $\Gamma_1(q) \backslash \mathrm{SL}(2, \mathbb{R})$, some $q \in \mathbb{Z}^+$.

If $y_g(T) \not\to 0$, $b_g(T) \to 0$: $\overline{\Gamma g u(\mathbb{R})}$ is 2-dimensional.

For μ-a.e. $g \in G$: $\Big(y_g(T)^{\frac{1}{4}} + b_g(T) \Big)^{\frac{1}{2}} \ll T^{-\frac{1}{8}(1-\varepsilon)}$ as $T \to \infty$.

Related results

Browning – Vinogradov ('16): For $\widetilde{u}(x) = \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \left(\frac{1}{2}x, \frac{1}{4}x^2\right) \right)$.

S – Vishe ('16): $\mathrm{SL}(2,\mathbb{R}) \ltimes (\mathbb{R}^2)^{\oplus k}$, special orbits of $u(x)$.

S – Södergren – Vishe (in progress) $\mathrm{SL}(2,\mathbb{R}) \ltimes (\mathbb{R}^2)^{\oplus k}$, general orbits of $u(x)$.

W. Kim ('21): $\mathrm{SL}(d,\mathbb{R}) \ltimes \mathbb{R}^d$ (expanding translates of lifts of horospheres in $\mathrm{SL}(d,\mathbb{R})$).

Lindenstrauss – Mohammadi – Wang ('22): $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{C})$.

Lei Yang ('22): Special unipotent orbits in $\mathrm{SL}(3,\mathbb{R})$.

Lindenstrauss – Mohammadi – Wang – Yang ('24): (Other) special unipotent orbits in $\mathrm{SL}(3,\mathbb{R})$.

Recall:

Theorem 1 (S, '15) For any $0 < y < 1$, $\vec{\xi} \in \mathbb{R}^2$, $\alpha < \beta$, $f \in \mathrm{C}_{\mathrm{b}}^8(X)$:

$$\left| \frac{1}{\beta - \alpha} \int_\alpha^\beta f\Big(\Gamma(1_2, \vec{\xi}\,)\, u(x)\, a(y)\Big)\, dx - \int_X f\, d\mu \right| \ll_\varepsilon \|f\|_{\mathrm{C}_{\mathrm{b}}^8} \frac{L}{\beta - \alpha} \Big(b_{\vec{\xi},L}(y) + y^{\frac{1}{4}} \Big)^{1-\varepsilon},$$

where $L = \max(1, |\alpha|, |\beta|)$ and

$$b_{\vec{\xi},L}(y) = \max_{q \in \mathbb{Z}^+} \min\left(\frac{1}{q^2}, \frac{\sqrt{y}}{Lq\langle q\xi_1\rangle}, \frac{\sqrt{y}}{q\langle q\xi_2\rangle} \right).$$

(Here $\langle \cdot \rangle$ =distance to nearest integer.)

Outline of proof of Theorem 1

Preliminary step: Replace "$\int_\alpha^\beta \cdots dx$" by a smooth integral "$\int_{\mathbb{R}} \cdots \nu(x)\, dx$".

Theorem 1' Fix $0 < \eta < 1$ and $\varepsilon > 0$.
Then for any $f \in \mathrm{C}_{\mathrm{b}}^8(\Gamma\backslash G)$, $\nu \in \mathrm{C}_c^2(\mathbb{R})$, $\vec{\xi} \in \mathbb{R}^2$, $0 < y < 1$:

$$\int_{\mathbb{R}} f\Big(\Gamma\,(1_2, \vec{\xi})\, u(x)\, a(y)\Big)\, \nu(x)\, dx = \int_{\Gamma\backslash G} f\, d\mu \int_{\mathbb{R}} \nu\, dx$$

$$+ O_{\eta,\varepsilon}\Big\{ \|f\|_{\mathrm{C}_{\mathrm{b}}^8} \|\nu\|_{\mathrm{W}^{1,1}}^{1-\eta} \|\nu\|_{\mathrm{W}^{2,1}}^{\eta}\, y^{\frac14} \log(1+y^{-1}) + \|f\|_{\mathrm{C}_{\mathrm{b}}^4} L \|\nu\|_{\mathrm{L}^\infty} \big(b_{\vec{\xi},L}(y) + y^{\frac14}\big)^{1-\varepsilon} \Big\},$$

where L is the smallest real number ≥ 1 such that $\mathrm{supp}(\nu) \subset [-L, L]$.

Outline of proof of Theorem 1
Step 1: Fourier decomposition in the torus $\mathrm{T}^2 = \mathbb{Z}^2\backslash\mathbb{R}^2$

Given $f \in \mathrm{C}_\mathrm{b}^8(X)$; view f as a left Γ-invariant function on G.

$$f((1_2, \vec{\xi})M) = f((1_2, \vec{\xi} + \vec{n})M), \quad \forall \vec{\xi} \in \mathbb{R}^2,\ M \in \mathrm{SL}(2,\mathbb{R}),\ \vec{n} \in \mathbb{Z}^2,$$

viz., "$\vec{\xi}$ lives in $\mathrm{T}^2 = \mathbb{Z}^2\backslash\mathbb{R}^2$".

Fourier expand w.r.t. $\vec{\xi}$:

$$\boxed{f((1_2, \vec{\xi})M) = \sum_{\vec{m}\in\mathbb{Z}^2} \widehat{f}(M, \vec{m}) e(\vec{m}\cdot\vec{\xi})},$$

with $\widehat{f}(M, \vec{m}) := \int_{\mathrm{T}^2} f((1_2, \vec{\xi})M) e(-\vec{m}\cdot\vec{\xi})\, d\vec{\xi}. \quad (M \in \mathrm{SL}(2,\mathbb{R}),\ \vec{m} \in \mathbb{Z}^2).$

$$\boxed{e(x) = e^{2\pi i x}}$$

The Γ-invariance of f implies: $\boxed{\widehat{f}(TM, \vec{m}) \equiv \widehat{f}(M, \vec{m}\,{}^{\mathrm{t}}T^{-1}), \quad \forall T \in \mathrm{SL}(2,\mathbb{Z}).}$

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The orbits of $\mathrm{SL}(2,\mathbb{Z})$ acting on $\mathbb{Z}^2$ by $\vec{m} \mapsto {}^{\mathrm{t}}T\vec{m}$ are:

$$\{\vec{0}\}, \quad \text{and} \quad \Big\{(m_1, m_2) \,:\, \gcd(m_1, m_2) = n\Big\} \quad \text{for} \;\; n = 1, 2, 3, \ldots .$$

Hence: Set $\boxed{\widetilde{f}_n := \widehat{f}\,(\cdot, (n, 0))}$ for $n \in \mathbb{Z}_{\geq 0}$; these determine *all* $\widehat{f}\,(\cdot, \vec{m})$!

$\widetilde{f}_0$ is left $\mathrm{SL}(2,\mathbb{Z})$-invariant, i.e. "lives on $Y = \mathrm{SL}(2,\mathbb{Z}) \backslash \mathrm{SL}(2,\mathbb{R})$"

$\forall n \geq 1:$ $\quad \widetilde{f}_n$ is left $\begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix}$-invariant, i.e. "lives on $\begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} \backslash \mathrm{SL}(2,\mathbb{R})$".

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Set $\boxed{\widetilde{f}_n(M) := \widehat{f}(M, (n, 0))}$ for $n \in \mathbb{Z}_{\geq 0}$ and $M \in \mathrm{SL}(2, \mathbb{R})$.

Now the Fourier expansion $f((1_2, \vec{\xi})M) = \sum_{\vec{m} \in \mathbb{Z}^2} \widehat{f}(M, \vec{m}) e(\vec{m} \cdot \vec{\xi})$ is collected into:

$$\boxed{f((1_2, \vec{\xi})M) = \widetilde{f}_0(M) + \sum_{n=1}^{\infty} \sum_{(c,d) \in \widehat{\mathbb{Z}}^2} \widetilde{f}_n\left(\begin{pmatrix} * & * \\ c & d \end{pmatrix} M \right) e(n(d\xi_1 - c\xi_2)),}$$

where $\widehat{\mathbb{Z}}^2$ is the set of *primitive* vectors in $\mathbb{Z}^2$, and $\begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$ (any choice).

Outline of proof of Theorem 1

Step 2: Evaluate 'horocycle integral' in terms of $\widetilde{f}_0, \widetilde{f}_1, \widetilde{f}_2, \ldots$.

Our integral:

$$\int_{\mathbb{R}} f\left(\Gamma\,(1_2, \vec{\xi})\,u(x)\,a(y)\right)\,\nu(x)\,dx$$

$$= \int_{\mathbb{R}} \widetilde{f}_0\left(\begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}\right)\,\nu(x)\,dx$$

$$+ \sum_{n=1}^{\infty} \sum_{\left(\begin{smallmatrix} c \\ d \end{smallmatrix}\right) \in \widehat{\mathbb{Z}}^2} e\big(n(d\xi_1 - c\xi_2)\big) \int_{\mathbb{R}} \widetilde{f}_n\left(\begin{pmatrix} * & * \\ c & d \end{pmatrix}\begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}\right)\,\nu(x)\,dx.$$

Recall that $\widetilde{f}_0$ lives on $Y = \mathrm{SL}(2,\mathbb{Z})\backslash\,\mathrm{SL}(2,\mathbb{R})$. Hence:

$$\int_{\mathbb{R}} \widetilde{f}_0\big(\cdots\big)\,\nu(x)\,dx = \underbrace{\int_Y \widetilde{f}_0\,d\mu_Y}_{\boxed{= \int_X f\,d\mu}} \int_{\mathbb{R}} \nu\,dx + O\Big(\big(\|\nu\|_{\mathrm{L}^1} + \|\nu'\|_{\mathrm{L}^1}\big)\,\|f\|_{\mathrm{C}_b^4}\,y^{\frac{1}{2}-\varepsilon}\Big).$$

(cf. Flaminio & Forni, '03 or S, '13).

Outline of proof of Theorem 1
Step 2: Evaluate 'horocycle integral' in terms of $\widetilde{f}_0, \widetilde{f}_1, \widetilde{f}_2, \ldots$.

Remains to bound:

$$\sum_{n=1}^{\infty} \sum_{\left(\begin{smallmatrix} c \\ d \end{smallmatrix}\right) \in \widehat{\mathbb{Z}}^2} e\big(n(d\xi_1 - c\xi_2)\big) \int_{\mathbb{R}} \widetilde{f}_n\left(\begin{pmatrix} * & * \\ c & d \end{pmatrix} \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}\right) \nu(x)\, dx.$$

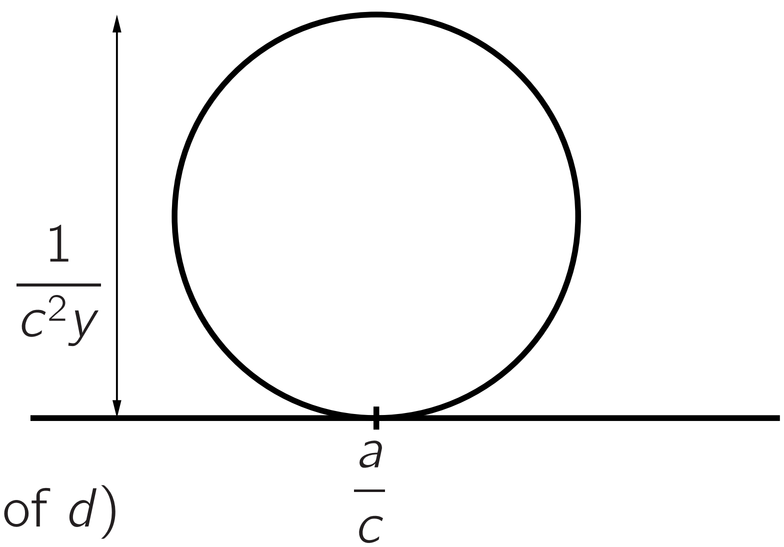
Recall that $\widetilde{f}_n$ lives on $\mathcal{M} := \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} \backslash \mathrm{SL}(2,\mathbb{R})$;

write $\widetilde{f}_n(u, v, \phi) := \widetilde{f}_n\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{v} & 0 \\ 0 & 1/\sqrt{v} \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\right)$

with $u \in \mathbb{R}/\mathbb{Z}$, $v > 0$, $\phi \in \mathbb{R}/2\pi\mathbb{Z}$.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \{x + iy \ : \ x \in \mathbb{R}\}$ for $c \neq 0$,

in the u, v-plane:

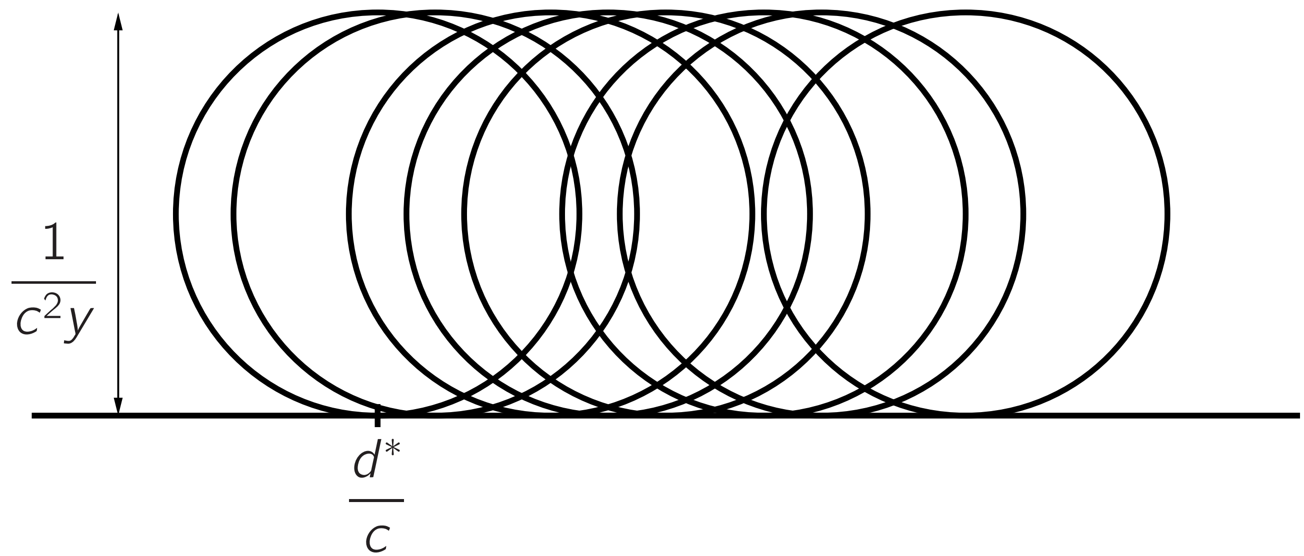


(here $a \equiv d^* \bmod c$, a multiplicative inverse of d)

Outline of proof of Theorem 1
Step 2: Evaluate 'horocycle integral' in terms of $\widetilde{f_0}, \widetilde{f_1}, \widetilde{f_2}, \ldots$.

For given n, $c \geq 1$, wish to bound:

$$\sum_{\substack{d \in \mathbb{Z} \\ (d,c)=1}} e(nd\xi_1) \int_{\mathbb{R}} \widetilde{f_n}\left(\begin{pmatrix} * & * \\ c & d \end{pmatrix} \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}\right) \nu(x)\, dx$$

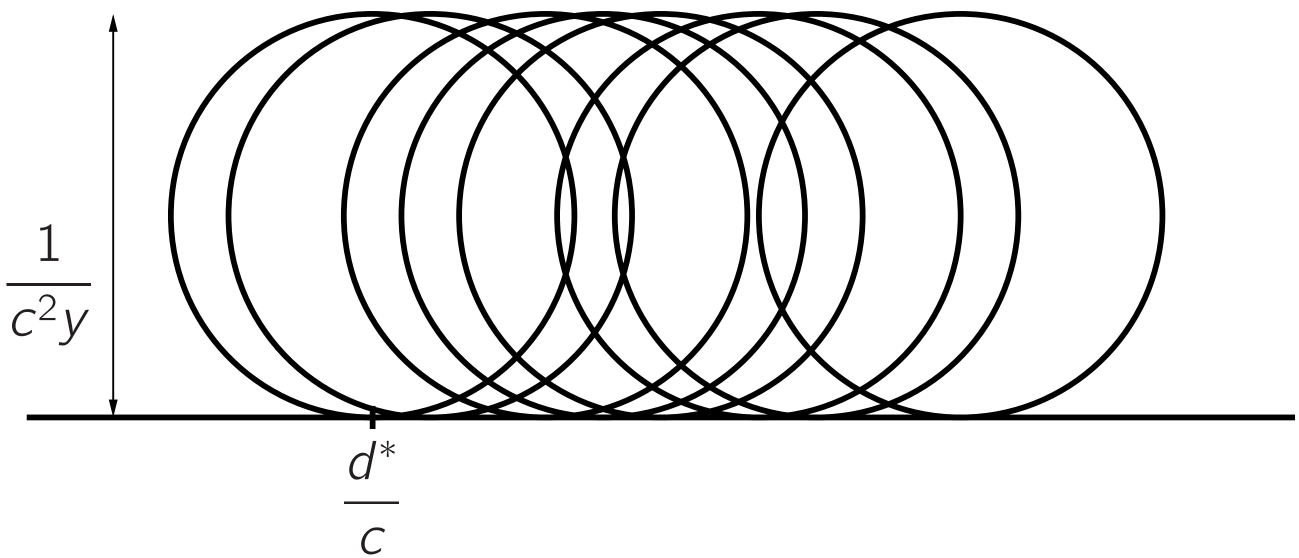


Outline of proof of Theorem 1
Step 2: Evaluate 'horocycle integral' in terms of $\widetilde{f}_0, \widetilde{f}_1, \widetilde{f}_2, \ldots$.

For given $n, c \geq 1$, wish to bound:

$$\sum_{\substack{d \in \mathbb{Z} \\ (d,c)=1}} e(nd\xi_1) \underbrace{\int_{\mathbb{R}} \widetilde{f}_n\left(\begin{pmatrix} * & * \\ c & d \end{pmatrix}\begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}\right) \nu(x)\, dx}$$

$$\boxed{= \int_0^{\pi} \widetilde{f}_n\left(\frac{d^*}{c} - \frac{\sin 2\phi}{2c^2 y}, \frac{\sin^2 \phi}{c^2 y}, \phi\right) \nu\left(-\frac{d}{c} + y \cot \phi\right) \frac{y\, d\phi}{\sin^2 \phi}}$$

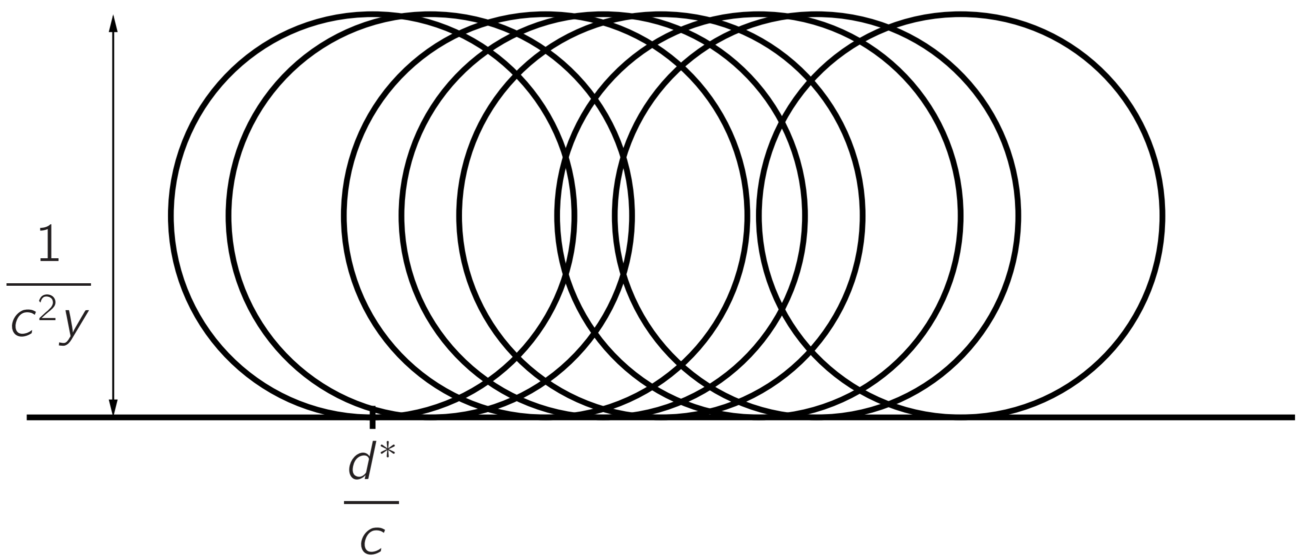


Outline of proof of Theorem 1
Step 2: Evaluate 'horocycle integral' in terms of $\widetilde{f_0}, \widetilde{f_1}, \widetilde{f_2}, \ldots$.

For given n, $c \geq 1$, wish to bound:

$$\sum_{\substack{d \in \mathbb{Z} \\ (d,c)=1}} e\big(nd\xi_1\big) \int_{\mathbb{R}} \widetilde{f_n}\left(\begin{pmatrix} * & * \\ c & d \end{pmatrix} \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}\right) \nu(x)\, dx$$

$$= \int_0^{\pi} \sum_{\substack{d \in \mathbb{Z} \\ (d,c)=1}} e\big(dn\xi_1\big) \widetilde{f_n}\left(\frac{d^*}{c} - \frac{\sin 2\phi}{2c^2 y}, \frac{\sin^2 \phi}{c^2 y}, \phi\right) \nu\left(-\frac{d}{c} + y \cot \phi\right) \frac{y\, d\phi}{\sin^2 \phi}$$



Outline of proof of Theorem 1
Step 2: Evaluate 'horocycle integral' in terms of $\widetilde{f}_0, \widetilde{f}_1, \widetilde{f}_2, \ldots$.

For given n, $c \geq 1$, wish to bound:

$$\sum_{\substack{d \in \mathbb{Z} \\ (d,c)=1}} e\big(dn\xi_1\big) \int_{\mathbb{R}} \widetilde{f}_n\left(\begin{pmatrix} * & * \\ c & d \end{pmatrix}\begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}\right) \nu(x)\, dx$$

$$= \int_0^{\pi} \sum_{\substack{d \in \mathbb{Z} \\ (d,c)=1}} \nu\Big(\underbrace{-\frac{d}{c}} + y \cot\phi\Big) \widetilde{f}_n\Big(\underbrace{\frac{d^*}{c}} - \frac{\sin 2\phi}{2c^2 y}, \frac{\sin^2\phi}{c^2 y}, \phi\Big) \underbrace{e\big(dn\xi_1\big)} \frac{y\, d\phi}{\sin^2\phi}$$

Hence, the task is to bound
$$\boxed{\sum_{\substack{d \in \mathbb{Z} \\ (c,d)=1}} g_1\Big(\frac{d}{c}\Big) g_2\Big(\frac{d^*}{c}\Big) e(dn\xi_1)},$$

for $g_1 \in \mathrm{C}_c^2(\mathbb{R})$, $g_2 \in \mathrm{C}^2(\mathbb{R}/\mathbb{Z})$, and $c \geq 1$ large.

Outline of proof of Theorem 1
Step 3: Proving cancellation in the sum

For $g_1 \in C_c^2(\mathbb{R})$, $g_2 \in C^2(\mathbb{R}/\mathbb{Z})$, $c \geq 1$ large:

$$\sum_{\substack{d\in\mathbb{Z}\\(c,d)=1}} g_1\Big(\frac{d}{c}\Big) g_2\Big(\frac{d^*}{c}\Big) e(dn\xi_1)$$

$$= \sum_{d\in(\mathbb{Z}/c\mathbb{Z})^\times} \underbrace{\left(\sum_{m\in\mathbb{Z}} g_1\Big(\frac{d}{c}+m\Big) e\Big(c\Big(\frac{d}{c}+m\Big)n\xi_1\Big)\right)}_{\boxed{\text{say } = \sum_{j\in\mathbb{Z}} a_j e\left(j\frac{d}{c}\right)}} \times \underbrace{g_2\Big(\frac{d^*}{c}\Big)}_{\boxed{= \sum_{k\in\mathbb{Z}} b_k e\left(k\frac{d^*}{c}\right)}}$$

$$= \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} a_j b_k S(j,k;c),$$

where $S(j,k;c)$ is the Kloosterman sum,

$$\boxed{S(j,k;c) := \sum_{d\in(\mathbb{Z}/c\mathbb{Z})^\times} e\Big(j\frac{d}{c} + k\frac{d^*}{c}\Big)}$$

Outline of proof of Theorem 1
Step 3: Proving cancellation in the sum

Using Weil's bound, $\left|S(n,m;c)\right| \leq \sigma(c)\sqrt{\gcd(n,m,c)}\sqrt{c}$ get:

$$\sum_{\substack{d\in\mathbb{Z}\\(c,d)=1}} g_1\left(\frac{d}{c}\right) g_2\left(\frac{d^*}{c}\right) e(dn\xi_1)$$

$$\ll \left(\|g_1\|_{L^1} + \|g_1''\|_{L^1}\right)\left\{\|g_2\|_{L^1} \sum_{k\in\mathbb{Z}} \frac{(c, \lfloor cn\xi_1 + k\rfloor)}{1+k^2} + \|g_2''\|_{L^1}\,\sigma(c)\sqrt{c}\right\}.$$

Outline of proof of Theorem 1
Step 3: Proving cancellation in the sum

Using Weil's bound, $\left|S(n,m;c)\right| \le \sigma(c)\sqrt{\gcd(n,m,c)}\sqrt{c}$ get:

$$\sum_{\substack{d\in\mathbb{Z}\\(c,d)=1}} g_1\left(\frac{d}{c}\right) g_2\left(\frac{d^*}{c}\right) e(dn\xi_1)$$

$$\ll \left(\|g_1\|_{L^1} + \|g_1''\|_{L^1}\right)\left\{\|g_2\|_{L^1}\sum_{k\in\mathbb{Z}}\frac{(c,\lfloor cn\xi_1 + k\rfloor)}{1+k^2} + \|g_2''\|_{L^1}\sigma(c)\sqrt{c}\right\}.$$

Apply to $\displaystyle\sum_{\substack{d\in\mathbb{Z}\\(d,c)=1}} \nu\left(\underbrace{-\frac{d}{c}} + y\cot\phi\right)\widetilde{f}_n\left(\underbrace{\frac{d^*}{c}} - \frac{\sin 2\phi}{2c^2 y}, \frac{\sin^2\phi}{c^2 y}, \phi\right)\underbrace{e(dn\xi_1)}.$

Use also: $\displaystyle\left|\left(\frac{\partial}{\partial u}\right)^{k_1}\left(\frac{\partial}{\partial v}\right)^{k_2}\left(\frac{\partial}{\partial \phi}\right)^{k_3}\widetilde{f}_n(u,v,\phi)\right| \ll_{m,k} \|f\|_{C_b^{m+k}} n^{-m} v^{\frac{m}{2}-k_1-k_2}.$

Outline of proof of Theorem 1
Step 3: Proving cancellation in the sum

For ξ_1 *of Diophantine type* $K \geq 2$ (viz., $\left|\xi_1 - \frac{p}{q}\right| \gg q^{-K}$, $\forall$q$\in \mathbb{Z}^+, p \in \mathbb{Z}$), we conclude:

$$\int_{\mathbb{R}} f\left(\Gamma\left(1_2, \vec{\xi}\right) u(x) \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix}\right) \nu(x)\, dx = \int_{\Gamma\backslash G} f\, d\mu \int_{\mathbb{R}} \nu\, dx$$

$$+ O_{\eta,\varepsilon,\xi_1}\left(\|f\|_{\mathrm{C}_{\mathrm{b}}^8} \|\nu\|_{\mathrm{W}^{1,1}}^{1-\eta} \|\nu\|_{\mathrm{W}^{2,1}}^{\eta}\right) y^{-\varepsilon+\min(\frac{1}{4},\frac{1}{K})}.$$

Outline of proof of Theorem 1
Step 4: The case of $n\xi_1$ near rational

For $n\xi_1$ (near) *rational*, $\sum_{\substack{d\in\mathbb{Z}\\(c,d)=1}} g_1\left(\frac{d}{c}\right) g_2\left(\frac{d^*}{c}\right) e(dn\xi_1)$ is *not* small!

Simplest example: If $n\xi_1 \in \mathbb{Z}$ then

$$\sum_{\substack{d\in\mathbb{Z}\\(c,d)=1}} g_1\left(\frac{d}{c}\right) g_2\left(\frac{d^*}{c}\right) e(dn\xi_1) = \varphi(c)\left(\int_{\mathbb{R}} g_1\right)\left(\int_{\mathbb{R}/\mathbb{Z}} g_2\right) + O\Big(\|g_1\|_{\mathrm{L}^1} + \|g_1''\|_{\mathrm{L}^1}\Big)\Big(\|g_2\|_{\mathrm{L}^1} + \|g_2''\|_{\mathrm{L}^1}\Big) c^{\frac{1}{2}+\varepsilon}.$$

Similarly pick up "main term(s)" whenever $n\xi_1$ is near rational!

Outline of proof of Theorem 1
Step 4: The case of $n\xi_1$ near rational

For $n\xi_1$ (near) *rational*, $\displaystyle\sum_{\substack{d\in\mathbb{Z}\\(c,d)=1}} g_1\Big(\frac{d}{c}\Big) g_2\Big(\frac{d^*}{c}\Big) e(dn\xi_1)$ is *not* small!

Then use also the summation over c, and the factor "$e(-cn\xi_2)$"!

$$\sum_{c=1}^{\infty} e(-cn\xi_2) \underbrace{\sum_{\substack{d\in\mathbb{Z}\\(d,c)=1}} e\big(dn\xi_1\big) \widetilde{f}_n\Big(\frac{d^*}{c} - \frac{\sin 2\phi}{2c^2 y}, \frac{\sin^2\phi}{c^2 y}, \phi\Big)\, \nu\Big(-\frac{d}{c} + y\cot\phi\Big)}$$

(Simplest example: If $n\xi_1 \in \mathbb{Z}$ then $\rightsquigarrow \sum_{c=1}^{\infty} e(-cn\xi_2)\varphi(c)$.)

. . . . $\square$