Counterexamples to generalizations of the Erdős $B + B + t$ problem

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Abstract

Confirming a conjecture of Erdős from the 1970s, Kra, Moreira, Richter, and Robertson proved that every subset of the integers with positive density contains a shifted sumset

$$B \oplus B + t = \{b_1 + b_2 + t : b_1, b_2 \in B, b_1 \neq b_2\}$$

for an infinite set $B \subseteq \mathbb{Z}$ and $t \geq \mathbb{Z}$. They then conjectured that the same result should be true in any (countable, discrete) abelian group and posed a number of related questions about other infinite sumset configurations in the integers.

We will give a negative answer to several of the questions and conjectures of Kra, Moreira, Richter, and Robertson by producing families of counterexamples based on a construction of Ernst Straus. Included among our counterexamples, we will exhibit, for any $\varepsilon > 0$, a set $A \subseteq \mathbb{N}$ with multiplicative upper Banach density at least $1 - \varepsilon$ such that A does not contain any dilated product set $\{b_1 b_2 t : b_1, b_2 \in B, b_1 \neq b_2\}$ for an infinite set $B \subseteq \mathbb{N}$ and $t \in \mathbb{Q}_{>0}$. We will also construct a set $A \subseteq \mathbb{N}$ with additive upper Banach density at least $1 - \varepsilon$ such that A does not contain any polynomial configuration $\{b_1^2 + b_2 + t : b_1, b_2 \in B, b_1 < b_2\}$ for an infinite set $B \subseteq \mathbb{N}$ and $t \in \mathbb{Z}$.