BOUNDS ON THE PYTHAGORAS NUMBER AND INDECOMPOSABLES IN BIQUADRATIC FIELDS

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Suppose that $\mathcal{O}$ is a commutative ring. By the set $\sum \mathcal{O}^2$, we mean the subset of elements in $\mathcal{O}$ which can be written as the sum of squares of elements belonging to $\mathcal{O}$. Similarly, the set $\sum^m \mathcal{O}^2 \subseteq \sum \mathcal{O}^2$ restricts to those elements which can be expressed as the sum of at most m squares. Then, the Pythagoras number of $\mathcal{O}$ is

$$\mathcal{P}(\mathcal{O}) = \inf \Big\{ m \in \mathbb{N} \cup \{+\infty\};\ \sum \mathcal{O}^2 = \sum^m \mathcal{O}^2 \Big\}.$$

We have quite a lot of information about its value if $\mathcal{O}$ is a field, which is, however, not true if $\mathcal{O} \subseteq \mathcal{O}_K$ is an order in a totally real number field K with the ring of algebraic integers $\mathcal{O}_K$. In that case, we know that $\mathcal{P}(\mathcal{O}) \in \mathbb{N}$, and we have an upper bound on $\mathcal{P}(\mathcal{O})$ depending only on the degree of K. On the other hand, the Pythagoras number of such orders can attain arbitrarily large values, and its value was determined for all orders in real quadratic fields. Besides that, only some partial results are known for fields of degrees 3 and 4.

In this talk, we will look more closely at the case of real biquadratic fields. For them, we will refine some bounds on the Pythagoras number of the entire ring of algebraic integers $\mathcal{O}_K$. We will also discuss the related question of universal quadratic forms over $\mathcal{O}_K$.

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